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Distributed Consensus Kalman Filtering Over Time-Varying Graphs ⋆

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Abstract: This paper proposes an improved method for distributed consensus Kalman filtering (DCKF). We introduce a minor modification to the consensus kalman filter proposed in (Olfati-Saber (2009)). Namely an extra averaging term is introduced into the filter update equations. In this direction, we propose a decentralized consensus gain that can be computed by each agent in the sensor network, and depends only on local properties of the network, i.e., the number of neighbors of each sensor. Moreover we prove that this scheme is stable for networks with time varying communication regime. Our results are compared to other existing solutions in the literature with a numerical example.

Keywords: Sensor networks, Estimation and filtering, Distributed control and estimation, Cooperative systems, distributed Kalman filtering

1. INTRODUCTION

Sensor networks comprise a group of agents equipped with communicating and sensing capabilities, which enable them to solve the problem of detecting a physical process utilizing cooperative sensing and estimation. This complex problem has been a major subject of interest in various research communities due to its wide range of applications including agriculture (Ojha et al. (2015)), security and surveillance (Wood et al. (2006); Onur et al. (2007)), health monitoring (Milenković et al. (2006)) and space research (Sun et al. (2005)).

Multi-agent cooperative estimation of some globally observable process is one of the most fundamental challenges in sensor networks (Bethke et al. (2007)). The networked system aims to obtain an estimate that globally converges to the true process state while considering constraints such as computational loads, the amount of shared data, and the overall system performance. A common solution for this challenge is that each agent in the system activates, in a distributed manner, an estimator which relies on local measurements of the process fused with the estimates from other agents in the network.

The authors of (Garin and Schenato (2010)) developed a tool to solve this problem by introducing a consensus-based term fused with a classical state estimator structure. This simple yet effective mechanism allowed the consideration of neighboring information in the individual agent estimation process. For example, the consensus H_∞ estimator is discussed in (Ugrinovskii (2011)), the consensus based distributed particle filtering as presented in (Hlinka et al. (2014)), and a consensus Kalman filter was formulated in (Olfati-Saber (2007); Alighanbari and How (2008)).

Discussing the consensus Kalman filter, one can witness increasing interest in recent years as new works are occupying different aspects of the filter. In (Deshmukh et al. (2017)), the consensus Kalman estimator proposed in (Olfati-Saber (2007)) was adapted, and they then derived the solution for both Kalman and consensus gains that will minimize the local mean squared error (MSE). Additionally, they compared simulation results with the sub-optimal solution suggested in (Olfati-Saber (2009)). The work (Wang et al. (2019)) utilized the same sub-optimal solution to derive a consensus extended Kalman filter in order to solve a spacecraft network relative motion estimation problem. The authors in (Battilotti et al. (2020)) made another variation on the sub-optimal consensus Kalman filter discussed in (Olfati-Saber (2009)) to solve the extended problem of networks with agents that have limited or null measurement capabilities.

The building block for most of these recent papers are based on the pioneering work conducted by Olfati-Saber in (Olfati-Saber (2009)) where he suggested a Kalman-like estimator with an additional consensus component. The consensus gain proposed in (Olfati-Saber (2009)) has the property that it can obtain very small values over time, rendering the consensus term contribution (i.e., the cooperative component of the filter) insignificant. In this case the estimator behaves more like a non-cooperative local Kalman filter (NCLKF) (i.e., each agent performs a Kalman filter with no additional information from neighbors).

In this direction, we proposed in (Priel and Zelazo (2021)) an alternative method for determining the consensus gain for the filter. With this method we derived for each time-step the maximal value of the consensus gain factor for which stability of the estimator is ensured. This guarantees that the consensus term, encouraging the agreement of estimates between neighboring agents, plays a nontrivial role in the estimator dynamics. Additionally, we proposed a decentralized consensus gain filter computation and proved stability for the homogeneous case where all agents have the same measurement model and noise characteristics.

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The area of decentralized consensus Kalman filtering is relatively new and has gained attention as there are few studies that focus on this type of estimator. However, many of these studies, such as (Kamal et al. (2013) and Battistelli et al. (2014)), have imposed on the communication topology some strong restrictions such as time-invariant and connected data flow. These restrictions are well described in Wang and Ren (2017).

Our contribution in this paper begins with proposing a decentralized consensus Kalman filter scheme that provides an upper bound on the error covariance estimation. In this architecture, the consensus gain is based on local network properties and thus can be implemented in systems with switching or time-varying communication networks without requiring any manual adaptation. Additionally, contrary to our earlier work (Priel and Zelazo (2021)), we prove the stability of this filter for the heterogeneous case where agents may not have the same measurement model. Finally, we present, via numerical example, superiority in performance of our proposed filter over the NCLKF and other consensus Kalman estimators appearing in the literature.

The paper is organized as follows. Section 2 provides an overview of the consensus Kalman filter estimator. In Section 3, a decentralized consensus scheme is proposed along with its stability analysis. In Section 4 simulation results are presented and finally, concluding notes are made in Section 5.

Notations: Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ the set of $n \times m$ real matrices. Let $\text{diag}\{m_i\}_{i=1}^n$ denote the block diagonal $nd \times nd$ matrix where the i^{th} block is equal to $m_i \in \mathbb{R}^{d\times d}$, and $\text{col}\{m_i\}_{i=1}^n$ denote the column stack of the vectors or matrices m_i. Let $[M]_{ij}$ denote the ij-entry of the matrix M. The maximal and minimal singular value of the matrix M are denoted by $\lambda_{max}(M)$ and $\lambda_{min}(M)$, respectively. The Frobenius norm of the matrix M is denoted as $|M|_F$.

2. THE CONSENSUS KALMAN ESTIMATOR

In this section we review the basic setup for constructing a distributed consensus Kalman filter along with reviewing existing sub-optimal solutions from the literature. Consider a network comprising N interacting agents where the interaction topology can be described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Here, $\mathcal{V} = \{1, 2, \ldots, N\}$ denotes the set of agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the edge set indicating which agents can exchange information with each other. The neighborhood of a node $v \in \mathcal{V}$ is the set of agents incident to it, i.e., $\mathcal{N}_v = \{u \in \mathcal{V} \,|\, (u, v) \in \mathcal{E}\}$. The graph can also be represented using the symmetric Laplacian matrix, $L \in \mathbb{R}^{N\times N}$ (Anderson Jr and Morley (1985)).

Each agent observes a linear discrete-time stochastic process described by the dynamics

$$\mathcal{P} : x_{k+1} = Ax_k + Bw_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector and w_k is an additive white Gaussian noise such that $\mathbb{E}[w_k w_l] = Q\delta_{kl}$, where δ_{kl} is the Dirac Delta function.

Each agent is capable of measuring the process state using the observation model

$$z_k^i = H^i x_k + v_k^i, \tag{2}$$

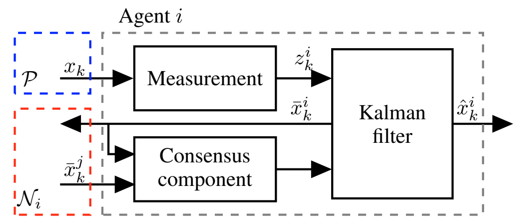


Fig. 1. DCKE structure for the i^{th} agent.

where $z_k^i \in \mathbb{R}^{m_i}$ is the measurement obtained by agent i, $H^i \in \mathbb{R}^{m_i\times n}$ is the observation matrix, and $v_k^i \in \mathbb{R}^{m_i}$ is a measurement noise assumed to also be additive white Gaussian noise with $\mathbb{E}\left[v_k^i v_l^i\right] = R^i\delta_{kl}$. Additionally we assume that A and $R^i \in \mathbb{R}^{m_i\times m_i}$ are invertible and that $[A, H^i]$ make an observable pair for every agent such that the noiseless NCLKF is asymptotically stable.

Olfati-Saber in (Olfati-Saber et al. (2007)) was the first to propose the distributed *Consensus Kalman* estimator (DCKE):

$$\hat{x}_k^i = \bar{x}_k^i + K_k^i\left(z_k^i - H^i\bar{x}_k^i\right) + C_k^i\sum_{j\in\mathcal{N}_i}\left(\bar{x}_k^j - \bar{x}_k^i\right), \tag{3}$$

where K^i and C^i are the Kalman and consensus gains of the i^{th} agent, respectively, and $\hat{x}_k^i$ and $\bar{x}_k^i$ are the *posteriori* and *a priori* state estimate of the i^{th} agent, respectively. The Kalman-Consensus estimator (3) is composed of a classic Kalman estimator term and a consensus term based on neighbors estimates, as illustrated in Figure 1.

In (Olfati-Saber (2009)), a distributed sub-optimal Kalman scheme was derived, which utilizes only (one hop) neighboring state estimates and discards any cross correlation terms that would have been incorporated in the optimal solution (Deshmukh et al. (2017)). The distributed sub-optimal Kalman is constructed as such:

$$\begin{cases} \textbf{Estimation} \\ K_k^i = P_k^i H^{i^T}\left(R^i + H^i\bar{P}_k^i H^{i^T}\right)^{-1} \\ \hat{P}_k^i = F_k^i\bar{P}_k^i F_k^{i^T} + K_k^i R^i K_k^{i^T} \\ \hat{x}_k^i = \bar{x}_k^i + K_k^i\left(z_k^i - H^i\bar{x}_k^i\right) + C_k^i\sum_{j\in\mathcal{N}_i}\left(\bar{x}_k^j - \bar{x}_k^i\right) \\ \textbf{Prediction} \\ \bar{x}_{k+1}^i = A\hat{x}_k^i \\ \bar{P}_{k+1}^i = A\hat{P}_k^i A^T + BQB^T, \end{cases} \tag{4}$$

here $\bar{P}_k^i$ is the i^{th} agent *a priori* error covariance and $\bar{P}_k^{j,i}$ is the i^{th} and j^{th} agent's *a priori* cross correlation term, and $F_k^i = I - K_k^i H^i$. The omission of the consensus terms from the Kalman gain and error covariance update equation is justified with the assumption that the consensus gain is relatively small. We have showed in (Priel and Zelazo (2021)) that special attention is required while selecting a small consensus gain since this might lead the consensus component to be negligible. We now wish to explore an alternate sub-optimal CKF along with a consensus gain which is computed in a distributed fashion.

3. DECENTRALIZED CONSENSUS FILTER AND GAIN DETERMINATION

In our previous work (Priel and Zelazo (2021)), we presented an approach for finding a consensus gain for the DCKE based on semi-definite programming. This calculation, however, must be done in a centralized manner, and the gain should be implemented for each agent in the sensor network. Note that any changes in the network structure, noise properties, or other, would require solving an SDP again, making this approach fragile in large-scale networks. These points motivate an alternative method for finding a suitable consensus factor that does not require any centralized computation. In this direction, we proposed in ((Priel and Zelazo (2021))) a decentralized approach for finding a suitable consensus gain which depends only on the local properties of the network for each agent. In this way, we can handle time-varying graphs as well. However, the stability of this solution was proven only for the homogeneous case where each agent holds the same observation model as its neighbors. We now extend these results to deal with heterogeneous sensing models for each agent.

Our solution begins with proposing a minor modification to the sub-optimal consensus Kalman filter structure (4). First, we assume the graph may be time-varying. We require no additional assumptions on the graph such as joint connectivity on finite time-intervals. An additional local averaging term is added to the predicted error covariance update equation as described here:

$$\begin{cases} \textbf{Estimation} \\ K_k^i = \bar{P}_k^i H^{i^T}\left(R^i + H^i \bar{P}_k^i H^{i^T}\right)^{-1} \\ \hat{P}_k^i = F_k^i \bar{P}_k^i F_k^{i^T} + K_k^i R^i K_k^{i^T} \\ \hat{x}_k^i = \bar{x}_k^i + K_k^i\left(z_k^i - H^i \bar{x}_k^i\right) + C_k^i \sum_{j\in\mathcal{N}_i}\left(\bar{x}_k^j - \bar{x}_k^i\right) \\ \textbf{Prediction} \\ \bar{x}_{k+1}^i = A\hat{x}_k^i \\ \bar{P}_{k+1}^i = \frac{1}{|\mathcal{N}_{i,k}|+1} A \sum_{j\in\mathcal{N}_{i,k}\cup\{i\}} \hat{P}_k^j A^T + BQB^T, \end{cases} \tag{5}$$

where $\mathcal{N}_{i,k}$ denotes the neighborhood of agent i at time step k. We consider the following choice of decentralized consensus gain,

$$C_k^i = \begin{cases} \frac{1}{|\mathcal{N}_{i,k}|+1} F_k^i & |\mathcal{N}_{i,k}| > 0 \\ 0 & |\mathcal{N}_{i,k}| = 0 \end{cases}, \tag{6}$$

Note that if the neighborhood set of an agent is empty (non-cooperative case) then the local filter is degraded into the classic Kalman filter for which the stability of the error dynamics is ensured.

Before proceeding to the stability analysis of (5), first let us establish some basic principles. We modify the original definition of *uniform detectability* as constructed in (Anderson and Moore (1981)) to handle a network of estimators:

Definition 1 (Absolute Uniform Detectability). *System* (1) *is said to be* absolutely uniformly detectable *by a sensor network modeled by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, if $\forall i \in \mathcal{V}$ there exist integers $m, t \geq 0$, and constants d, γ^i with $0 \leq d < 1$, $0 \leq \gamma^i < \infty$ such that whenever*

$$||\phi_{k+t,k}\zeta|| \geq d||\zeta|| \tag{7}$$

for some ζ and k, then

$$\zeta^T G_{k-m,k}^i \zeta \geq \gamma^i \zeta^T \zeta, \tag{8}$$

where

$$G_{k-m,k}^i = \sum_{l=k-m}^{k} \phi_{l,k}^T H^{i^T} (R^i)^{-1} H^i \phi_{l,k}, \tag{9}$$

with $\phi_{k,k} = I_n$, $\phi_{l,k} = \phi_{l,l+1}\phi_{l+1,l+2}...\phi_{k-1,k}$ and $\phi_{k-1,k} = A_{k-1}^{-1} = A^{-1}$.

Additionally, for future development, we lay out the following assumption and lemmas.

Assumption 1. *For every $i \in \mathcal{V}$ there exist real positive constants $\bar{a}$, $\underline{a}$, $\bar{h}^i$, $\underline{h}^i$, $\bar{q}$, $\underline{q}$, $\bar{r}^i$, $\underline{r}^i$, $\bar{b}^i$, $\underline{b}^i$ such that the following bounds are fulfilled:*

$$\underline{a}^2 I_n \leq AA^T \leq \bar{a}^2 I_n \tag{10}$$

$$H^i H^{i^T} \leq (\bar{h}^i)^2 I_{m_i} \tag{11}$$

$$\underline{q} I_f \leq Q \leq \bar{q} I_f \tag{12}$$

$$\underline{r}^i I_{m_i} \leq R^i \leq \bar{r}^i I_{m_i} \tag{13}$$

$$\underline{b}^2 I_n \leq BB^T \leq \bar{b}^2 I_n. \tag{14}$$

Assumption 1 shall be used later on to prove that the inverse error covariance is lower bounded. From an application point of view these assumption are reasonable as most problems deal with bounded dynamics and noise properties.

Lemma 1. *If there exists two positive scalars $\underline{p}^i$ and $\bar{p}^i$ such that $\underline{p}^i I_n \leq \hat{P}_k^i \leq \bar{p}^i I_n$ and $\hat{P}_k^i > 0, \forall i \in \mathcal{V}$, then there always exists a strictly positive real number*

$$\alpha_k^i = \frac{\underline{p}^i}{\sup_{j\in\mathcal{N}_i} \bar{p}^j},$$

such that

$$\frac{1}{|\mathcal{N}_{i,k}|+1} \sum_{j\in\mathcal{N}_i\cup\{i\}} \hat{P}_k^j \leq \frac{1}{\alpha_k^i} \hat{P}_k^i.$$

Proof. Given that error covariance for each agent is positive definite, we have that $\hat{P}_k^j \leq \frac{\bar{p}^j}{\underline{p}^i} \hat{P}_k^i$, therefore

$$\begin{aligned} \frac{1}{|\mathcal{N}_{i,k}|+1} \sum_{j\in\mathcal{N}_{i,k}\cup\{i\}} \hat{P}_k^j &\leq \frac{1}{|\mathcal{N}_{i,k}|+1} \sum_{j\in\mathcal{N}_{i,k}\cup\{i\}} \frac{\bar{p}^j}{\underline{p}^i} \hat{P}_k^i \\ &\leq \frac{1}{|\mathcal{N}_{i,k}|+1} \frac{\sup_{j\in\mathcal{N}_i} \bar{p}^j}{\underline{p}^i} \sum_{j\in\mathcal{N}_{i,k}\cup\{i\}} \hat{P}_k^i \\ &\leq \frac{1}{\alpha_k^i} \hat{P}_k^i. \end{aligned} \tag{15}$$

Lemma 2. *[(Li et al. (2016))] Consider* (5) *under Assumption 1, then for every $k > 0$, there always exists a strictly positive real number*

$$\xi = \left(1 + \bar{b}^2 \bar{q}\left(\frac{\underline{r} + \underline{b}^2 \underline{q} \bar{h}^2}{\underline{a}^2 \underline{r} \underline{q} \underline{b}^2}\right)\right)^{-1} < 1,$$

such that

$$(\bar{P}_{k+1}^i)^{-1} \geq \xi A^{-T} \left(\frac{1}{|\mathcal{N}_{i,k}|+1} \sum_{j\in\mathcal{N}_{i,k}\cup\{i\}} \hat{P}_k^j\right)^{-1} A^{-1}, \forall i \in \mathcal{V}. \tag{16}$$

We are now ready to present our main result with the following theorem.

Theorem 1 (Decentralized DCKE Stability). *Consider a group of N agents interacting over a directed time-varying graph $\mathcal{G}_k$. Each agent observes the process model* (1), *which is assumed* *to be absolutely uniformly detectable. Each agent employs the observation model* (2). *Then the noiseless error dynamics with the consensus Kalman filter* (5) *and the choice of consensus gain* (6) *are asymptotically stable.*

Proof. We divide our proof into 2 parts where in the first, we show that the error covariance estimate $\hat{P}_k^i$ is upper bounded using the observability Gramian. In the second part we construct a Lyapunov function that is monotonically decreasing along the system trajectories, and show that it is radially unbounded using the error covariance estimate $\hat{P}_k^i$.

By Lemma 2 we know that

$$\begin{aligned}(\bar{P}_{k+1}^i)^{-1} &= \left(\frac{1}{|\mathcal{N}_{i,k}|+1} A \sum_{j\in\mathcal{N}_{i,k}\cup\{i\}} \hat{P}_k^i A^T + BQB^T\right)^{-1} \\ &\geq \xi A^{-T}\left(\frac{1}{|\mathcal{N}_{i,k}|+1}\sum_{j\in\mathcal{N}_{i,k}\cup\{i\}}\hat{P}_k^j\right)^{-1} A^{-1}.\end{aligned} \tag{17}$$

Additionally, by Lemma 1 it holds that

$$\frac{1}{|\mathcal{N}_{i,k}|+1}\sum_{j\in\mathcal{N}_i\cup\{i\}}\hat{P}_{k-1}^j \leq \frac{1}{\alpha_k^i}\hat{P}_{k-1}^i$$

such that inequality (17) can be further simplified:

$$(\bar{P}_k^i)^{-1} \geq \alpha_k^i \xi A^{-T}(\hat{P}_{k-1}^i)^{-1}A^{-1}. \tag{18}$$

Using the Woodbury inversion lemma we have that

$$\begin{aligned}(\hat{P}_k^i)^{-1} &= (F_k^i\bar{P}_k^i)^{-1} \\ &= (\bar{P}_k^i - \bar{P}_k^i H^{i^T}(H^i\bar{P}_k^iH^{i^T} + R^i)^{-1}\bar{P}_k^i)^{-1} \\ &= (\bar{P}_k^i)^{-1} + H^{i^T}R^{i^{-1}}H^i.\end{aligned} \tag{19}$$

Plugging (18) into (19) yields:

$$\begin{aligned}(\hat{P}_k^i)^{-1} &\geq \alpha_{k-1}^i\xi A^{-T}(\hat{P}_{k-1}^i)^{-1}A^{-1} + H^{i^T}R^{i^{-1}}H^i \\ &\geq \alpha_{k-2}^i\alpha_{k-1}^i\xi^2A^{-T}A^{-T}(\hat{P}_{k-2}^i)^{-1}A^{-1}A^{-1} \\ &+ \alpha_{k-1}^i\xi A^{-T}H^{i^T}R^{i^{-1}}H^iA^{-1} + H^{i^T}R^{i^{-1}}H^i \\ &\geq \xi^k\phi_{0,k}^T(\hat{P}_0^i)^{-1}\phi_{0,k} + \sum_{l=k-1}^{0}\alpha_l^i\xi^{k-l}\phi_{k,l}^TH^{i^T}R^{i^{-1}}H^i\phi_{k,l} \\ &> \beta^i\gamma^iI_n,\end{aligned} \tag{20}$$

where $\beta^i = \inf_{l\in 1,2\ldots k-1}\alpha_l^i\xi^{k-l}$, thus it shown the inverse error covariance is lower bounded. We now proceed to the second part of the proof where we show that the error dynamics is asymptotically stable. To this, first we formulate the local error dynamics:

$$\begin{aligned}\eta_k^i &= F_k^iA\eta_{k-1}^i + \frac{1}{|\mathcal{N}_{i,k}|+1}F_k^iA\sum_{j\in\mathcal{N}_{j,k}}\left[\eta_{k-1}^j - \eta_{k-1}^i\right] \\ &+ K_k^iv_k^i + F_k^iB\omega_k \\ &= \frac{1}{|\mathcal{N}_{i,k}|+1}F_k^iA\sum_{j\in\mathcal{N}_{i,k},i}\eta_{k-1}^j + K_k^iv_k^i + F_k^iB\omega_k,\end{aligned} \tag{21}$$

and the augmented noiseless error dynamics are

$$\begin{aligned}\eta_k &= \mathrm{diag}\{F_k^iA\}_{i=1}^N\left(I_{Nn} - \left(\mathcal{D}_k^{-1}L_k\otimes I_n\right)\right)\eta_{k-1} \\ &= \mathrm{diag}\{F_k^i\}_{i=1}^N((I_N - \mathcal{D}_k^{-1}L_k)\otimes A)\eta_{k-1},\end{aligned} \tag{22}$$

with $\mathcal{D}_k = \mathrm{diag}\{|\mathcal{N}_{i,k}|+1\}_{i=1}^N$. For simplicity, we denote $\tilde{F}_k = \mathrm{diag}\{F_k^i\}_{i=1}^N$, $\tilde{A} = (I_N\otimes A)$ and $\tilde{L}_k = (I_N - \mathcal{D}_k^{-1}L_k)\otimes I_n$ such that the error dynamics can expressed as

$$\eta_k = \tilde{F}_k\tilde{A}\tilde{L}_k\eta_{k-1}. \tag{23}$$

It is immediate that for the non-cooperative case, i.e, when $\tilde{L}_k = I_{Nn}$ we obtain the noiseless NCLKF error dynamics.

We continue with defining the following Lyapunov function,

$$V_k = \eta_k^T\Xi_k^{-1}\eta_k^T,$$

Where Ξ_k is defined with the recursion,

$$\begin{aligned}\Xi_0 &= \mathrm{diag}\{\hat{P}_0^i\}_{i=1}^N \\ \Xi_k &= \tilde{F}_k\tilde{A}\tilde{L}_k\Xi_{k-1}\tilde{L}_k^T\tilde{A}^T\tilde{F}_k^T + \tilde{Q}_k.\end{aligned} \tag{24}$$

It was shown in our previous work (Priel and Zelazo (2021)) that the step Lyapunov difference function is negative definite. We are now left to show that the matrix Ξ_k^{-1} is lower bounded. To do so we construct the local error covariance:

$$\begin{aligned}\mathbb{E}\left[\eta_k^i\eta_k^{i^T}\right] &= F_k^iA\mathbb{E}\left[\frac{\left(\sum_{j\in\mathcal{N}_i\cup\{i\}}\eta_{k-1}^j\right)\left(\sum_{j\in\mathcal{N}_i\cup\{i\}}\eta_{k-1}^j\right)^T}{(|\mathcal{N}_{i,k}|+1)^2}\right]A^TF_k^{i^T} \\ &+ K_k^i\mathbb{E}\left[v_k^iv_k^{i^T}\right]K_k^{i^T} + F_k^iB\mathbb{E}\left[w_k^iw_k^{i^T}\right]B^TF_k^{i^T}.\end{aligned} \tag{25}$$

We know that for any 2 vectors $X, Y \in \mathbb{R}^m$ the following inequality holds:

$$XY^T + YX^T \leq XX^T + YY^T,$$

then

$$\begin{aligned}\mathbb{E}\left[\eta_k^i\eta_k^{i^T}\right] &\leq F_k^iA\mathbb{E}\left[\frac{\left(\sum_{j\in\mathcal{N}_i\cup\{i\}}\eta_{k-1}^j\eta_{k-1}^{jT}\right)}{|\mathcal{N}_{i,k}|+1}\right]A^TF_k^{i^T} \\ &+ K_k^i\mathbb{E}\left[v_k^iv_k^{i^T}\right]K_k^{i^T} + F_k^iB\mathbb{E}\left[w_kw_k^T\right]B^TF_k^{i^T}.\end{aligned} \tag{26}$$

The augmented error covariance is

$$\begin{aligned}\mathbb{E}\left[\eta_k\eta_k^T\right] &= \tilde{F}_k\tilde{A}\tilde{L}_k\mathbb{E}\left[\eta_{k-1}\eta_{k-1}^T\right]\tilde{L}_k^T\tilde{A}^T\tilde{F}_k^T \\ &+ \mathrm{diag}\{K_k^iR^iK_k^{i^T} + F_k^iBQB^TF_k^{i^T}\}_{i=1}^N\end{aligned} \tag{27}$$

Therefore if in (24) we choose $\tilde{Q}_k = \mathrm{diag}\{K_k^iR^iK_k^i + BQB^T\}_{i=1}^N$, where recall that $\mathbb{E}[w_kw_l] = Q\delta_{kl}$. Then we obtain $\Xi_k = \mathbb{E}\left[\eta_k\eta_k^T\right]$ and the inequality in (26) can be rephrased as:

$$\Xi_k^{ii} \leq F_k^iA\frac{\sum_{j\in\mathcal{N}_i\cup\{i\}}\Xi_{k-1}^{jj}}{|\mathcal{N}_{i,k}|+1}A^TF_k^{i^T} + \tilde{Q}_k, \tag{28}$$

and since $\Xi_0 = \hat{P}_0$, this yields:

$$\Xi_k^{ii} \leq F_k^iA\frac{\sum_{j\in\mathcal{N}_i\cup\{i\}}\hat{P}_{k-1}^j}{|\mathcal{N}_{i,k}|+1}A^TF_k^{i^T} + \tilde{Q}_k = \hat{P}_k^i. \tag{29}$$

To this point we have showed that the diagonal elements of the matrix Ξ_k are upper bounded, which implies that $\mathrm{trace}\,(\Xi_k)$ is upper bounded as well. For any symmetric positive definite matrix P , the following holds,

$$\lambda_{max}(P) \leq \sum_{l=1}^n\lambda_l(P) = \mathrm{trace}\,(P), \tag{30}$$

suggesting that Ξ_k is upper bounded (and Ξ_k^{-1} is lower bounded), this completes the proof. □

Remark 1. *While the requirement for $[A, H^i]$ to form an observable pair for each agent may appear stringent, it enables* *the relaxation of conditions related to network connectivity. In contrast, the authors of (Battistelli et al. (2014)) assumed collective observability (i.e., $[A, \mathrm{col}\{H^i\}_{i=1}^N]$ make an observable pair), however they assumed a static and connected communication graph.*

We have showed that the consensus Kalman filter (5) with the choice of consensus gain (6) are stable for the networks which may change over time. In contrast, (Sandell and Olfati-Saber (2008)) proposed the following decentralized consensus gain,

$$C_k^i = \frac{\epsilon}{1 + |\hat{P}_k^i|_F} \hat{P}_k^i, \tag{31}$$

where ϵ is some predetermined constant. The design constant ϵ can be pre-calibrated, however no mid-run modification techniques were provided for this constant in case of, for example, a change in the graph structure.

4. SIMULATION RESULTS

The following numerical example was taken from (Olfati-Saber et al. (2007)) with minor modifications. Consider a robot performing a noisy "snail" trajectory with the following dynamics,

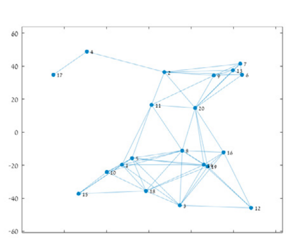
$$x_{k+1} = \underbrace{\begin{bmatrix} 0.9996 & -0.0283 \\ 0.0283 & 0.9996 \end{bmatrix}}_{A} x_k + \underbrace{0.375 \cdot I_2}_{B} w_k. \tag{32}$$

The initial state vector and the covariance matrix for each agent are set to be $x_0^i = [15, -10]^T, P_0^i = 10I$, respectively. Additionally, the process noise covariance is $Q = I_2$. A network of 20 sensors are randomly positioned in some field of interest (see Figure 2a) where a communication link between 2 sensors exists only if their distance is below some threshold (< 40 meters). Furthermore, each agent with an even number measure the robot's y-axis position while the agents with an odd number measure its x-axis position such that:

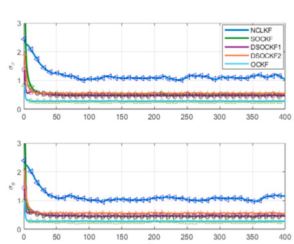
$$H^i = \begin{cases} [1, 0] & i \in \{1, 3, ..., 19\} \\ [0, 1] & i \in \{2, 4, ..., 20\} \end{cases}. \tag{33}$$

The measurement noise covariance for agent k is $R^k = \sqrt{k}$.

Although, $[A, H^i]$ make an observable pair for each individual sensor, the observability is relatively weak for the non-measured axis, i.e., while the robot is in transition between quadrants one would expect a relatively large estimation error since the position in one axis hardly vary while the position in the other can vary significantly.



(a) A sensor network of 20 agents.



(b) Standard deviation of the agents' state estimation for both axes, comparison between 5 distributed state estimators over 100 Monte-Carlo runs.

Fig. 2. Communication topology and state estimation standard deviation.

We provide a comparison between 5 state estimators:

NCLKF: the non-cooperative local Kalman filter with null consensus gain;

SOCKF: the sub-optimal consensus Kalman filter with a centralized consensus factor as presented in (Priel and Zelazo (2021));

DSOCKF1: the decentralized sub-optimal consensus Kalman filter with $\epsilon = 0.1$ and consensus gain (31);

DSOCKF2: the decentralized sub-optimal consensus Kalman filter (5) with consensus gain (6).

OCKF: the optimal consensus Kalman filter as derived in (Deshmukh et al. (2017)).

The compared performance measures are twofold: the agents state estimation standard deviation (Fig. 2b) calculated as

$$\sigma^x = \frac{1}{MC} \sum_{j=1}^{MC} \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} \left([1 \ 0]\, \hat{x}^{i,j} - \frac{1}{N} [1 \ 0] \sum_{i=1}^{N} \hat{x}^{i,j} \right)^2}$$

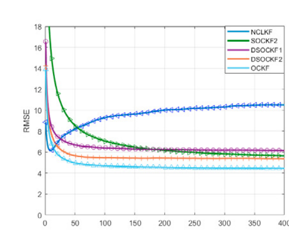
$$\sigma^y = \frac{1}{MC} \sum_{j=1}^{MC} \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} \left([0 \ 1]\, \hat{x}^{i,j} - \frac{1}{N} [0 \ 1] \sum_{i=1}^{N} \hat{x}^{i,j} \right)^2},$$

where MC denotes the number of Monte-Carlo runs and $\hat{x}^{i,j}$ is the ith agent state estimation for the jth run. The true averaged root mean squared error (Fig. 3a) calculated as

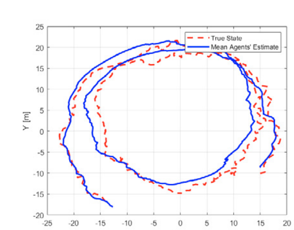
$$\mathrm{RMSE} = \frac{1}{MC} \sum_{j=1}^{MC} \sqrt{\sum_{i=1}^{N} (\mathbb{E}[(\eta^{i,j})^T \eta^{i,j}])},$$

where $\eta^{i,j} = \hat{x}^{i,j} - x$.

In Figure 2b one can observe the agents rate of convergence and stability of the agents' estimation error for all 5 distributed state estimators. As shown, all consensus Kalman filter obtain small state estimation standard deviation with respect to the NCLKF. Moreover, Figure 3a demonstrate the superiority in performance of the DSOCKF2 over DSOCKF1 and as expected, the OCKF outperformed all other estimators. What is perhaps most astonishing, is that DSOCKF2 outperform SOCKF as well. This result might be correlated to the similarity in structure of DSOCKF2 gain to that of OCKF as discussed in (Priel and Zelazo (2021)).



(a) Root mean squared error, comparison between 5 state estimators over 100 Monte-Carlo runs.



(b) Trajectory of the true state and the agents' mean estimate utilizing SOCKF2.

Fig. 3. Joint estimation performance.

Figure 3b illustrates the true and mean estimated trajectory of the robot using SOCKF2. As shown, the proposed filter provides good tracking results.

To conclude, we compare the robustness of the proposed decentralized consensus Kalman filter to the filter proposed in (Sandell and Olfati-Saber (2008)). To do so, we simulate a communication topology switch at each time instance by randomizing the maximal distance between 2 nodes to have a

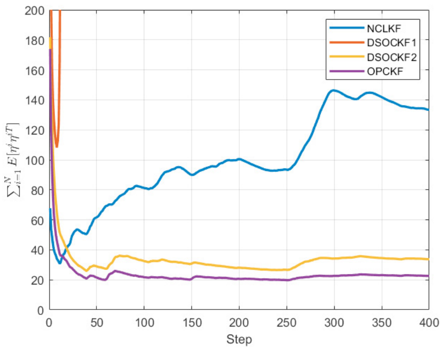


Fig. 4. Sum of all agents mean squared error for a constantly graph switching network.

communication link. The sum of all agents MSE are presented in Figure 4, where we compare between 4 estimators: NCLKF, DSOCKF1, and DSOCKF2 and OCKF. As shown, the DSOCKF1 becomes unstable after the first couple of switches, while the DSOCKF2 remains stable for the entire duration.

5. CONCLUSION

We have presented a widely common sub-optimal *consensus Kalman* filter scheme and presented a novel modification to the filter prediction formulation. This modification , paved way for constructing a consensus gain with which the estimation error stability is ensured. Moreover, the consensus gain is constructed in a decentralized manner which does not require global knowledge of graph properties. Finally we presented performance superiority of the filter over existing solution in the literature and over the non-cooperative local Kalman filter.

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