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An Improved Distributed Consensus Kalman Filter Design Approach

Aviv Priel and Daniel Zelazo

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***Abstract*— This paper proposes an improved design approach for distributed consensus Kalman filtering (DCKF). We provide an improved consensus gain factor compared to the sub-optimal design proposed in [1]. This factor is derived from an LMI appearing in the stability analysis of the DCKF and can be computed using semi-definite programming. We also propose a decentralized consensus gain that can be computed by each agent in the sensor network, and depends only on local properties of the network, i.e., the number of neighbors of each sensor. We show in simulation that this approach holds even for networks with time varying communication regime. Our results are compared to other existing solutions in the literature with a numerical example.**

I. INTRODUCTION

Sensor networks comprise a group of agents equipped with sensing devices and communicating capabilities in order to solve some common task such as cooperative sensing and estimation of a detectable physical process. This complex problem has been a major subject of interest in various research communities due to its wide range of applications including agriculture [2], security and surveillance [3], [4], health monitoring [5] and space research [6].

One of the fundamental challenges in sensor networks deal with cooperative estimation of some globally observable process [7], [8]. In this scenario, each agent in the system activates, in a distributed fashion, an estimator which relies on local measurements of the process fused with the estimates from other agents in the network. The networked system aims to globally converge to the true process state while considering constraints such as computational loads, the amount of shared data, and the overall system performance.

A recently developed tool to solve this problem is the introduction of a consensus-based term fused with a classical state estimator structure [9]. This provides a mechanism for accounting for neighboring information. For example, the consensus H_∞ estimator is discussed in [10], the consensus based distributed particle filtering as presented in [11], and a consensus Kalman filter was formulated in [12], [13]. In the consensus Kalman filter arena, one can witness increasing interest in recent years as new works are occupying different aspects of the filter. In [14], the consensus Kalman estimator proposed in [12] was adapted, and they then derived the solution for both Kalman and consensus gains that will minimize the local mean squared error (MSE). Additionally they have compared simulation results with the sub-optimal solution suggested in [1]. The work [15] utilized the same sub-optimal solution to derive a consensus extended Kalman filter in order to solve a spacecraft network relative motion estimation problem. The authors in [16] made another variation on the sub-optimal consensus Kalman filter discussed in [1] to solve the extended problem of networks with agents that have limited or null measurement capabilities.

These recent papers are all built on the pioneering work conducted by Olfati-Saber in [1] where he suggested a Kalman-like estimator with an additional consensus component. In this work, both optimal and sub-optimal consensus filter gains were derived, while guaranteeing the stability of the estimator. However, to our knowledge, no comparison has been made between this approach and a *non-cooperative local Kalman filter* (NCLKF), where each sensor implements a Kalman filter without any exchange of information from other sensors in the network. Furthermore, the selected consensus gain derived in [1] might obtain small values rendering the consensus term contribution insignificant. In this case the estimator behaves more like a NCLKF without agents reaching agreement on their estimates.

In this direction, our contribution begins with proposing an alternative method for determining the consensus gain for the filter. With this method, we derive for each time step the maximal value of the consensus gain factor for which stability of the estimator is ensured. This ensures that the consensus term, encouraging the agreement of estimates between neighboring agents, plays a nontrivial role in the estimator dynamics. Utilizing convex optimization techniques to extract this aforementioned consensus factor, we demonstrate through simulation examples the superiority in performance over the NCLKF and others consensus gains found in the literature. We also show the proposed estimator is mean square error Lyapunov stable. Additionally, we propose a decentralized consensus gain which is based on local network properties and thus, can be implemented in systems with switching or time-varying communication networks without requiring any manual adaptation. Once more, superiority in performance over the NCLKF and others are presented through simulations results.

The paper is organized as follows. Section II provides an overview of the consensus Kalman filter estimator. In Section III, we derive a semi-definite program to determine a stabilizing consensus gain factor. In addition, a decentralized consensus gain is proposed. In Section IV simulation results are presented and finally, concluding notes are made in Section V.

Notations: Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$

The authors are with the Faculty of Aerospace Engineering, Technion-Israel Institute of Technology, Haifa 32000, Israel (e-mail: savivpri@campus.technion.ac.il; dzelazo@technion.ac.il). This work was supported by Grant No. 2285/20 from the Israel Science Foundation and Grant No. 2017658 from the United States-Israel Binational Science Foundation (BSF).

the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ the set of $n\times m$ real matrices. Let $\text{diag}\{M^i\}_{i=1}^n$ denote the block diagonal $nd\times nd$ matrix where the i^{th} block is equal to $M^i\in\mathbb{R}^{d\times d}$, and let $[M]_{ij}$ denote the ij-entry of the matrix M. The maximal and minimal singular value of the matrix M are denoted by $\lambda_{max}(M)$ and $\lambda_{min}(M)$, respectively, while $\rho(M)$ denotes the spectral radius of M. The Frobenius norm of the matrix M is denoted as $|M|_F$.

II. The Consensus Kalman Estimator

In this section we review the basic setup for constructing a distributed consensus Kalman filter along with reviewing existing sub-optimal solutions from the literature. Consider a network comprising N interacting agents where the interaction topology can be described by an undirected graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$. Here, $\mathcal{V}=\{1,2,...,N\}$ denotes the set of agents and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ denotes the edge set indicating which agents can exchange information with each other. The neighborhood of a node $v\in\mathcal{V}$ is the set of agents incident to it, i.e., $\mathcal{N}_v=\{u\in\mathcal{V}\,|\,(u,v)\in\mathcal{E}\}$. The graph can also be represented using the symmetric Laplacian matrix, $L\in\mathbb{R}^{N\times N}$ [17].

Each agent observes a linear discrete-time stochastic process described by the dynamics

$$\mathcal{P}: x_{k+1}=Ax_k+Bw_k, \tag{1}$$

where $x_k\in\mathbb{R}^n$ is the state vector and w_k is an additive white Gaussian noise such that $\mathbb{E}\left[w_k w_l\right]=Q\delta_{kl}$, where δ_{kl} is the Dirac Delta function.

Each agent is capable of measuring the process state using the observation model

$$z_k^i=H^i x_k+v_k^i, \tag{2}$$

where $z_k^i\in\mathbb{R}^{m^i}$ is the measurement obtained by agent i, $H^i\in\mathbb{R}^{m^i\times n}$ is the observation matrix, and $v_k^i\in\mathbb{R}^{m^i}$ is a measurement noise assumed to also be additive white Gaussian noise with $\mathbb{E}\left[v_k^i v_l^i\right]=R^i\delta_{kl}$. Additionally we assume that $R^i\in\mathbb{R}^{m^i\times m^i}$ is invertible and that $[A,H^i]$ make an observable pair for every agent such that the noiseless NCLKF is asymptotically stable.

The distributed *Consensus Kalman* estimator (DCKE) was first proposed by [18] and is constructed as

$$\hat{x}_k^i=\bar{x}_k^i+K_k^i\left(z_k^i-H^i\bar{x}_k\right)+C_k^i\sum_{j\in\mathcal{N}_i}\left(\bar{x}_k^j-\bar{x}_k^i\right), \tag{3}$$

where K^i and C^i are the Kalman and consensus gains of the i^{th} agent, respectively, and $\hat{x}^i$ and $\bar{x}^i$ are the *posteriori* and *a priori* state estimate of the i^{th} agent, respectively. The Kalman-Consensus estimator (3) is composed out of a classic Kalman estimator term and a consensus term based on neighbors estimates as illustrated in Figure 1.

In [1], a distributed optimal Kalman gain was derived by minimizing the local MSE with respect to K_k^i. The optimal gain was found to be

$$K_k^i=\left(P_k^iH^{iT}+C_k^i\sum_{j\in\mathcal{N}_i}\left(\bar{P}_k^{j,i}-\bar{P}_k^i\right)H^{iT}\right)\left(R^i+H^i\bar{P}_k^iH^{iT}\right)^{-1}, \tag{4}$$

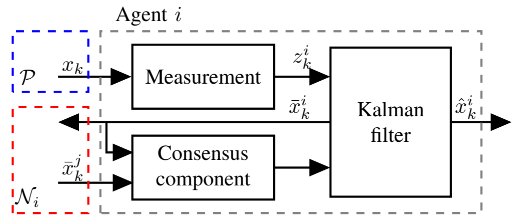


Fig. 1: DCKE structure for the i^{th} agent.

where $\bar{P}^i$ is the i^{th} agent *a priori* error covariance and $\bar{P}^{j,i}$ is the i^{th} and j^{th} agent's *a priori* cross correlation term. The corresponding update equations incorporate two-hop neighbors information exchange. For example, in a complete graph this would mean that each agent would retrieve $N(N-1)$ cross correlation terms at each step. The latter served as the motivation to construct the following sub-optimal distributed consensus Kalman filter (SOCKF) which utilizes only (one hop) neighboring state estimates and discards the consensus terms from the error covariance and Kalman gain update equations:

$$\begin{cases}\textbf{Prediction}\\ \quad\bar{x}_k^i=A\hat{x}_{k-1}^i\\ \quad\bar{P}_k^i=A\hat{P}_{k-1}^iA^T+BQB^T\\ \textbf{Estimation}\\ \quad K_k^i=P_k^iH^{iT}\left(R^i+H^i\bar{P}_k^iH^{iT}\right)^{-1}\\ \quad\hat{P}_k^i=F_k^i\bar{P}_k^iF_k^{iT}+K_k^iR^iK_k^{iT}\\ \quad\hat{x}_k^i=\bar{x}_k^i+K_k^i\left(z_k^i-H^i\bar{x}_k^i\right)+C_k^i\sum\limits_{j\in\mathcal{N}_i}\left(\bar{x}_k^j-\bar{x}_k^i\right),\end{cases} \tag{5}$$

where $F_k^i=I-K_k^iH^i$. The omission of the consensus terms from the Kalman gain and error covariance update equation is justified with the assumption that the consensus gain is relatively small. We would like to emphasize that one must be careful while selecting a small consensus gain since this might lead the consensus component in the DCKE to be negligible. Nevertheless, in [1] it was shown that (5) has stable estimator dynamics.

III. IMPROVED CONSENSUS GAIN SELECTION

In this section we explore both centralized and decentralized approaches for designing the consensus gain term C_k^i in (5).

A. Centralized Consensus Gain Determination

We propose a new consensus gain for the SOCKF update scheme (5). We aim to extract the maximal consensus gain in a manner that will ensure the stability of the local estimation error (and thus, for the sum of all errors as well).

Theorem 1 (DCKE Stability)**.** *Consider a group of N agents interacting over a connected graph $\mathcal{G}$ where each observes the process (1) with observation model* (2)*. The noiseless estimation error with the Kalman consensus filter (5) and the choice of consensus gain $C_k^i=\gamma_kP_k^iF_k^{iT^{-1}}$ is asymptotically* *stable for any $\gamma_k \in [0, \gamma_k^*] \forall k$. Furthermore, γ_k^* can be obtained as the maximum value for which*

$$\begin{aligned}\mathcal{K}_k(\gamma_k) &= \text{diag}\{\hat{P}_{k-1}^{i^{-1}} - A^T F_k^{iT} \hat{P}_k^{i^{-1}} F_k^i A\}_{i=1}^N \\ &- \gamma_k^2 \left(L \otimes A\right)^T \text{diag}\{F_k^{i^{-1}} \hat{P}_k^{i^{-1}} F_k^{iT^{-1}}\}_{i=1}^N \left(L \otimes A\right) \\ &+ 2\gamma_k \left(L \otimes A^T A\right),\end{aligned} \tag{6}$$

is positive semi-definite, and can be found using semi-definite programming.

Proof. The proof for this theorem follows the same line as presented in [1] with an additional section to establish the range of consensus gains γ_k. First we choose a quadratic Lyapunov function and show that for $\gamma_k = 0$, the Lyapunov function is monotonically decreasing. We then prove that there must be some γ_k^* such that for any $\gamma_k \in [0, \gamma_k^*]$, the Lyapunov function is monotonically not increasing.

Let $\eta_k = \hat{x}_k - x_k$ and $\bar{\eta}_k = \bar{x}_k - x_k$ be the estimation and prediction errors, respectively. The noiseless error dynamics are

$$\begin{aligned}\bar{\eta}_k^i &= A\eta_{k-1}^i, \\ \eta_k^i &= \underbrace{\left(I - K_k^i H^i\right)}_{F_k} \bar{\eta}_k^i + C_k^i \sum_{j \in \mathcal{N}_j} \left(\bar{\eta}_k^j - \bar{\eta}_k^i\right).\end{aligned}$$

Consider now the following Lyapunov function,

$$V_k = \sum_{i=1}^N \eta_k^{iT} \hat{P}_k^{i^{-1}} \eta_k^i. \tag{7}$$

The Lyapunov step difference function along the system trajectories is

$$\begin{aligned}\delta V_k = V_k - V_{k-1} &= \sum_{i=1}^N \eta_k^{iT} \hat{P}_k^{i^{-1}} \eta_k^i - \sum_{i=1}^N \eta_{k-1}^{iT} \hat{P}_{k-1}^{i^{-1}} \eta_{k-1}^i \\ &= \sum_{i=1}^N \left(F_k^i A\eta_{k-1}^i + C_k^i A u_k^i\right)^T \hat{P}_k^{i^{-1}} \left(F_k^i A\eta_{k-1}^i + C_k^i A u_k^i\right) - \eta_{k-1}^{iT} \hat{P}_{k-1}^{i^{-1}} \eta_{k-1}^i \\ &= \sum_{i=1}^N \eta_{k-1}^{iT} \left(A^T F_k^{iT} \hat{P}_k^{i^{-1}} F_k^i A - \hat{P}_{k-1}^{i^{-1}}\right) \eta_{k-1}^i \\ &+ 2\sum_{i=1}^N \bar{\eta}_k^{iT} F_k^{iT} \hat{P}_k^{i^{-1}} C_k^i u_k^i + \sum_{i=1}^N u_k^{iT} C_k^{iT} \hat{P}_k^{i^{-1}} C_k^i u_k^i,\end{aligned}$$

where

$$u_k^i = \sum_{j \in \mathcal{N}_i} \left(\bar{x}_k^j - \bar{x}_k^i\right) = \sum_{j \in \mathcal{N}_i} \left(\bar{\eta}_k^j - \bar{\eta}_k^i\right). \tag{8}$$

Let us consider only the term which is not dependent on the consensus gain, $\Psi_k^i = -\hat{P}_{k-1}^{i^{-1}} + A^T F_k^{iT} \hat{P}_k^{i^{-1}} F_k^i A$. Plugging in (5) into this expression produces

$$\Psi_k^i = -\hat{P}_{k-1}^{i^{-1}} + A^T F_k^{iT} (F_k^i A \hat{P}_{k-1}^i A^T F_k^{iT} + \Pi_k^i)^{-1} F_k^i A, \tag{9}$$

where $\Pi_k^i = K_k^i R^i K_k^{iT} + F_k^i Q F_k^{iT}$. Multiplying $\hat{P}_{k-1}^i$ on both sides of (9) yields

$$\begin{aligned}&\hat{P}_{k-1}^i \Psi_k^i \hat{P}_{k-1}^i = \\ &\hat{P}_{k-1}^i A^T F_k^{iT} \left(F_k^i A \hat{P}_{k-1}^i A^T F_k^{iT} + \Pi_k^i\right)^{-1} F_k^i A \hat{P}_{k-1}^i - \hat{P}_{k-1}^i.\end{aligned}$$

Utilizing the Woodbury matrix identity [19] (inversion lemma) and multiplying once more $\hat{P}_{k-1}^{i^{-1}}$ on both sides gives

$$\Psi_k^i = -\hat{P}_{k-1}^{i^{-1}} \left(\hat{P}_{k-1}^{i^{-1}} + A^T F_k^{iT} \Pi_k^{i^{-1}} F_k^i A\right) \hat{P}_{k-1}^{i^{-1}}. \tag{10}$$

Since $\Pi_k^{i^{-1}}$ and $\hat{P}_{k-1}^{i^{-1}}$ are positive definite, Ψ_k^i is negative definite. We are left to find a consensus gain such that δV_k shall always remain non-positive. Consider the consensus gain structure proposed by [1] of

$$C_k^i = \gamma_k \hat{P}_k^i \left(F_k^{iT}\right)^{-1} = \gamma_k \bar{P}_k^i. \tag{11}$$

Implementing (11) into δV_k produces

$$\delta V_k = \sum_{i=1}^N \eta_{k-1}^{iT} \Psi_k^i \eta_{k-1}^i + 2\gamma_k \sum_{i=1}^N \bar{\eta}_k^{iT} u_k^i + \gamma_k^2 \sum_{i=1}^N u_k^{iT} Y_k^i u_k^i, \tag{12}$$

where $Y_k = F_k^{T^{-1}} \hat{P}_k F_k^{-1}$. The second term in (12) can be simplified using the graph Laplacian and (8) as

$$2\gamma_k \sum_{i=1}^N \bar{\eta}_k^{iT} u_k^i = -2\gamma_k \eta_{k-1}^T \left(L \otimes A^T A\right) \eta_{k-1}, \tag{13}$$

where η_{k-1} is the augmented agents' estimation error vector at the $k-1$ step. It is immediate that the third term in (12) is positive semi definite:

$$\gamma_k^2 \sum_{i=1}^N u_k^{iT} Y_k^i u_k^i = \gamma_k^2 \eta_{k-1}^T \left(L \otimes A\right)^T Y_k \left(L \otimes A\right) \eta_{k-1},$$

with $Y_k = \text{diag}\{Y_k^i\}_{i=1}^N$. Therefore we can write:

$$\delta V_k = -\eta_{k-1}^T \mathcal{K}_k \eta_{k-1}, \tag{14}$$

with

$$\mathcal{K}_k = \left(-\Psi_k + 2\gamma_k \left(L \otimes A^T A\right) - \gamma_k^2 \left(L \otimes A\right)^T Y_k \left(L \otimes A\right)\right), \tag{15}$$

and $\Psi_k = \text{diag}\{\Psi_k^i\}_{i=1}^N$.

We showed in (10) that for $\gamma_k = 0$, corresponding to a NCLKF, $\mathcal{K}_k$ is positive definite. We now show that there must be a positive upper bound on γ_k for which $\mathcal{K}_k$ is positive sem-definite. In this direction, we recall Sylvester's criteria [20], which states that a matrix is positive definite if and only if its leading principle minors are all positive. In this direction, let $a_1 = -[\Psi_k]_{11}, a_2 = [2\left(L \otimes A^T A\right)]_{11}$ and $a_3 = -[(L \otimes A)^T Y_k (L \otimes A)]_{11}$. If $a_1 + a_2\gamma_k + a_3\gamma_k^2 < 0$ (the first leading principle minor of $\mathcal{K}_k$ is negative), then $\mathcal{K}_k$ is not positive semi-definite. Therefore, there must be some γ_k^* satisfying $0 < \gamma_k^* < \frac{-a_2 - \sqrt{a_2^2 - 4a_1 a_3}}{2a_3}$ for which the matrix $\mathcal{K}_k$ is positive semi-definite, and for any $\gamma_k \in [0, \gamma_k^*]$, $\mathcal{K}_k$ is positive definite and the noiseless error dynamic is asymptotically stable.

The next step in our proof is to find a method for extracting the consensus factor γ_k^*. Here, we employ the Schur complement lemma [21]. Let us consider the constraint $\mathcal{K}_k(\gamma_k) \succeq 0$, and observe

$$\begin{aligned}&\mathcal{K}_k(\gamma_k) = \\ &\left(\Psi_k + 2\gamma_k(L \otimes A^T A)\right) - (\gamma_k(L \otimes A)^T) Y_k (\gamma_k(L \otimes A)).\end{aligned}$$

Since $[A, H^i]$ make an observable pair for all agents, the matrix F_k^i is full rank. Additionally, we know that $\hat{P}_k^i$ is positive-definite, therefore Y_k and its inverse are positive-definite. Thus we can conclude that $K_k(\gamma_k) \succeq 0$ if and only if

$$\begin{bmatrix} \Psi_k + 2\gamma_k(L \otimes A^T A) & \gamma_k(L \otimes A)^T \\ \gamma_k(L \otimes A) & Y_k^{-1} \end{bmatrix} \succeq 0.$$

This is an LMI constraint in γ_k. We can then construct the semi-definite program

$$\begin{aligned} &\max_{\gamma_k} \gamma_k \\ &\texttt{s.t.} \begin{bmatrix} \Psi_k + 2\gamma_k(L \otimes A^T A) & \gamma_k(L \otimes A)^T \\ \gamma_k(L \otimes A) & Y_k^{-1} \end{bmatrix} \succeq 0, \end{aligned} \tag{16}$$

to obtain the largest value γ_k ensuring that $\mathcal{K}_k$ is positive definite. This completes the proof. □

We are now interested in comparing the consensus gain found from (16) with the gain proposed in [1]. The gain from [1] is given as

$$\gamma_k = \sqrt{\frac{\lambda_{min}(\Psi_k)}{\lambda_{max}((L \otimes A)Y_{k+1}(L \otimes A))}}. \tag{17}$$

Although this consensus factor selection holds, $\lambda_{min}(\Psi_k)$ can obtain small values prior to convergence and thus the contribution of the consensus component is mitigated prematurely (before agreement was secured). This motivates our reason for finding the largest possible consensus gain. A larger gain will ensure the consensus term in the update equation will provide a meaningful network-level contribution. We also demonstrate in Section IV that the DCKE with the gain (16) out performs the gain proposed in [1].

B. Decentralized Consensus Gain Determination

In the previous sub-section, we presented an approach for finding a consensus gain for the DCKE based on semi-definite programming. This calculation, however, must be done in a centralized manner, and the gain should be implemented for each agent in the sensor network. Note that any changes in the network structure, noise properties, or other, would require solving the SDP in (16) again making this approach fragile in large-scale networks. These points motivate an alternative method for finding a suitable consensus factor that does not require any centralized computation.

In this direction, we propose a decentralized approach for finding a suitable consensus gain that depends only on the local properties of the network for each agent. In this way, we can handle time-varying graphs as well.

Consider a group of N agents, interacting over a time-varying graph $\mathcal{G}_k$, which is assumed to be connected at each time-instant k. Each sensor observes the process (1) with observation model (2). Consider now the decentralized consensus gain,

$$C_k^i = \frac{1}{|\mathcal{N}_{i,k}|} F_k^i, \tag{18}$$

where $\mathcal{N}_{i,k}$ denotes the neighborhood of agent i at time step k. Then, the local noiseless error dynamics are

$$\begin{aligned} \eta_k^i &= F_k^i A \eta_{k-1}^i + \frac{1}{|\mathcal{N}_{i,k}|} F_k^i A \sum_{j \in \mathcal{N}_{j,k}} \left[\eta_{k-1}^j - \eta_{k-1}^i\right] \\ &= \frac{1}{|\mathcal{N}_{i,k}|} F_k A \sum_{j \in \mathcal{N}_{j,k}} \eta_{k-1}^j, \end{aligned} \tag{19}$$

and the augmented noiseless error dynamics are

$$\begin{aligned} \eta_k &= \operatorname{diag}\{F_k^i A\}_{i=1}^N \left(I_{Nn} - \left(\mathcal{D}_k^{-1} L_k \otimes I_n\right)\right) \eta_{k-1} \\ &= \operatorname{diag}\{F_k^i\}_{i=1}^N ((I_N - \mathcal{D}_k^{-1} L_K) \otimes A) \eta_{k-1}, \end{aligned} \tag{20}$$

with $\mathcal{D}_k = \operatorname{diag}\{|\mathcal{N}_{i,k}|\}_{i=1}^N$. It is immediate that for the non-cooperative case, i.e., when $L_k = 0$, we obtain the noiseless NCLKF error dynamics. Under the case where each sensor has the same observation of the process, we can arrive at the following result.

Proposition 1. *Assume that each sensor in the network measures the process* (1) *using the same observation model*

$$z_k^i = H x_k + v_k^i, i = 1, \ldots, N,$$

where v_k^i is the zero-mean Gaussian measurement noise with $\mathbb{E}[v_k^i v_l^i] = R\delta_{kl}$. Then the error dynamics (20) *are asymptotically stable.*

Proof. In the case where each sensor uses the same measurement model, it follows that $F_k^i = \bar{F}_k$ for all agents. The error dynamics can then be simplified to

$$\begin{aligned} \eta_k &= \operatorname{diag}\{F_k^i A\}_{i=1}^N \left(I_{Nn} - \left(\mathcal{D}_k^{-1} L_k \otimes I_n\right)\right) \eta_{k-1} \\ &= (I_N \otimes \bar{F}_k A)((I_N - \left(\mathcal{D}_k^{-1} L_k)\right) \otimes I_n\right) \eta_{k-1} \\ &= \left((I_N - (\mathcal{D}_k^{-1} L_k)) \otimes \bar{F}_k A\right) \eta_{k-1}. \end{aligned}$$

From the stability of the NCLKF, it follows that $\lim_{k\to\infty} \left(\prod_k \bar{F}_k A\right) = 0$.[1] Furthermore, the matrix $I_N - \left(\mathcal{D}_k^{-1} L_k\right))$ is row stochastic at each time step k, and thus its spectral radius is always unity, and in particular,

$$\rho\left(\lim_{k\to\infty}\left(\prod_k (I_N - (\mathcal{D}_k^{-1} L_k)\right)\right) = 1.$$

Therefore,

$$\lim_{k\to\infty} \eta_k = \lim_{k\to\infty}\left(\prod_k (I_N - (\mathcal{D}_k^{-1} L_k) \otimes \prod_k \bar{F}_k A\right) \eta_0 = 0.$$

□

The result of Proposition 1 may be restrictive, as we are assuming each sensor has the same measurement model with noise characteristics. On the other hand, such a model may be useful when employing a homogeneous sensor network and aiming for faster convergence of the estimate compared to using a single sensor. Currently, we do not have a proof for the general case of heterogeneous sensor measurements, however we note that in numerical simulation, over a variety

[1] Here we use an abuse of conventional notation and define $\prod_{k=1}^n M_k = M_n M_{n-1} \cdots M_2 M_1$.

of random network properties, the heterogeneous case gives promising results; we explore this in the next section.

The above proposition provides an extremely simple method to find a consensus factor that works. In contrast, [22] proposed the following decentralized consensus gain,

$$C_k^i = \frac{\epsilon}{1 + |\hat{P}_k^i|_F} \hat{P}_k^i, \tag{21}$$

where ϵ is some predetermined constant. The design constant ϵ can be pre-calibrated, however no mid-run modification techniques were provided for this constant in case of, for example, a change in the graph structure.

Remark 1. *It should be noted that the consensus gain structure in the centralized scheme of Theorem 1 is not the same as the one proposed in (18). The centralized consensus gain was chosen to ensure explicitly that the Lyapunov function (7) decreases along the system trajectories. On the other hand, the decentralized gain was chosen to simplify the structure of the error dynamics.*

Although not having the same structure, we would expect that the centralized consensus gain found using (16) would out perform the proposed decentralized consensus gain. This is due to the fact that the centralized estimator employs global network properties to compute the consensus gain, whereas, in the decentralized scheme, only local network properties are employed. For the numerical example presented in section IV we note that this is not the case, and in fact the decentralized scheme show better results.

IV. SIMULATION RESULTS

The following numerical example was taken from [18] with minor modifications. Consider a robot performing a noisy "snail" trajectory with the following dynamics,

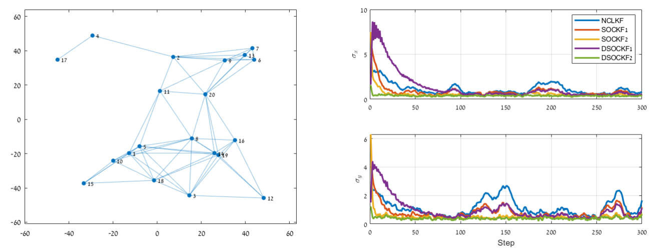
$$x_{k+1} = \underbrace{\begin{bmatrix} 0.9996 & -0.0283 \\ 0.0283 & 0.9996 \end{bmatrix}}_{A} x_k + \underbrace{0.375 \cdot I_2}_{B} w_k. \tag{22}$$

The initial state vector and the covariance matrix for each agent are set to be $x_0^i = [15, -10]^T, P_0^i = 10I$, respectively. Additionally, the process noise covariance is $Q = I_2$. A network of 20 sensors are randomly positioned in some field of interest (see Figure 2a) where a communication link between 2 sensors exists only if their distance is below some threshold (< 40 meters). Furthermore, each agent with an even number measure the robot's y-axis position while the agents with an odd number measure its x-axis position such that:

$$H^i = \begin{cases} [1, 0] & i \in \{1, 3, ..., 19\} \\ [0, 1] & i \in \{2, 4, ..., 20\} \end{cases}. \tag{23}$$

The measurement noise covariance for agent k is $R^k = \sqrt{k}$.

Although, $[A, H^i]$ make an observable pair for each individual sensor, the observability is relatively weak for the non-measured axis, i.e., while the robot is in transition between quadrants one would expect a relatively large estimation error since the position in one axis hardly vary while the position in the other can vary significantly.



(a) A sensor network of 20 agents.

(b) Standard deviation of the agents' state estimation.

Fig. 2: Communication topology and estimators STD.

We provide a comparison between 5 state estimators:

NCLKF: the non-cooperative local Kalman filter with null consensus gain;

SOCKF1: the sub-optimal consensus Kalman filter with consensus factor (17);

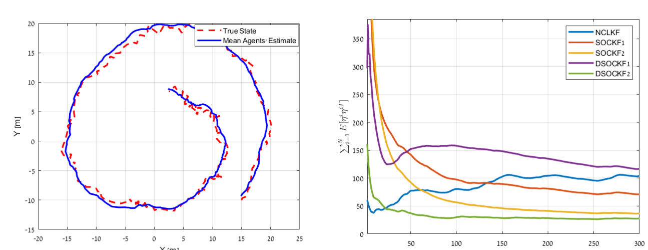
SOCKF2: the sub-optimal consensus Kalman filter with consensus factor (16) (computed utilizing CVX toolbox [23]);

DSOCKF1: the decentralized sub-optimal consensus Kalman filter with $\epsilon = 0.1$ and consensus gain (21);

DSOCKF2 the decentralized sub-optimal consensus Kalman filter with consensus gain (18).

The compared performance measures are twofold: the agents state estimation standard deviation (Figure 2b) and the true sum of the agents mean squared error $\sum_{i=1}^{N}(\mathbb{E}[\eta^{iT}\eta^i])$ (Figure 3b).

In Figure 2b one can observe the agents rate of convergence and stability of the agents' estimation error for all 5 state estimators. As shown, the SOCKF2 converges with the fastest rate among the centralized filters while maintaining a relatively constant state estimation standard deviation (even through quadrants transitions). Additionally, one can observe the similarity between SOCKF1 to the NCLKF estimator. This results due to the extremely small consensus factor gain used by SOCKF1 which effectively ignores the effect of the consensus component, thus turning SOCKF1 into a NCLKF estimator. In the decentralized schemes we observe superiority of the DSOCKF2 over DSOCKF1.



(a) Trajectory of the true state and the agents' mean estimate utilizing SOCKF2.

(b) Sum of MSE for all agents, comparison between 5 state estimators.

Fig. 3: Joint estimation performance.

Figure 3a illustrates the true and mean estimated trajectory of the robot using SOCKF2. As shown, the proposed filter provides good tracking results. Figure 3b further demon-

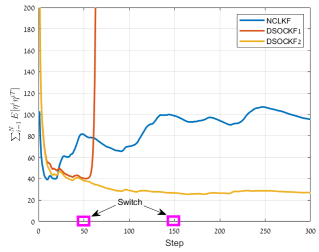


Fig. 4: Sum of all agents mean squared error. The communication graph switches at time step 50 and 150.

strates the superiority of SOCKF2 and DSOCK2 over the others by presenting the lowest mean squared error. This result is expected as there is more information for the agents to process. Once more it is shown that for this specific graph topology, the SOCKF1 shows an advantage over the NCLKF estimator while DSOCKF1 does not show any advantage over the NCLKF which means that its gain selection was poorly randomized. What is perhaps most astonishing is that these simulations indicate the decentralized consensus gain selection out performs the centralized gain. Understanding this performance improvement based on the different structure of the consensus gains is a subject of ongoing research.

To conclude, we compare the robustness of the proposed decentralized consensus Kalman filter to the filter proposed in [22]. To do so, we simulate a communication topology switch at 2 time instants - step 50 and step 150. The sum of all agents MSE are presented in Figure 4, where we compare between 3 estimators: NCLKF, DSOCKF1, and DSOCKF2. As shown, the DSOCKF1 becomes unstable after the first switch, while the DSOCKF2 remains stable for the entire duration.

V. CONCLUSIONS

We have presented a widely common sub-optimal *consensus Kalman* filter scheme and presented new solutions for determining the consensus gain. In the centralized scheme, we proposed a semi-definite program for extracting an upper bound on the consensus factor which does not affect the state estimation error stability. Additionally, we proposed a decentralized scheme which does not require global knowledge of graph properties. Finally we presented performance superiority of both schemes over existing solutions in the literature and over the non-cooperative local Kalman filter.

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