Pointing Consensus and Bearing-Based Solutions to the Fermat–Weber Location Problem

Minh Hoang Trinh, *Member, IEEE*, Daniel Zelazo, *Senior Member, IEEE*, and Hyo-Sung Ahn, *Senior Member, IEEE*

***Abstract*—The pointing consensus problem asks each agent in a multiagent system to agree on their headings toward a common target. This paper proposes a decentralized approach to the pointing consensus problem by simultaneously solving three smaller problems: bearing-only measurement based network localization, target decision, and heading coordination. The proposed solution guarantees that all agents' headings almost globally asymptotically target any weighted centroid of the agents' positions. Furthermore, based on this approach, two decentralized solutions for the Fermat–Weber location problem are proposed and analyzed. Simulation results are also provided to support the analysis.**

***Index Terms*—Bearing-only measurements, decentralized control, Fermat–Weber location problem (FWLP), multiagent systems, network localization.**

I. Introduction

In recent years, a lot of research interest has focused on multiagent systems thanks to their ubiquitous applications in civilian and military defenses. In this scheme, the consensus algorithm [1] has been extensively studied as a decentralized solution to coordinate a group of multiple agents. Given n agents having different initial state values, by exchanging and updating the states based on the weighted sum of differences, all agents' states eventually reach the same value [1], [2]. The states of the agents could be auxiliary variables used for decision and control tasks [3], [4], or physical variables such as positions, velocities, and attitudes in the formation control or attitude synchronization problem [5]–[7].

Unlike the usual consensus problem, the pointing consensus (or concurrent targeting) problem requires all agents in a group to direct their heading vectors toward a common point in space. This problem found applications in satellite formations [8], antenna arrays [9], and camera networks. For instance, pointing consensus is important in coordinating multiple collectors and combiner spacecrafts in synthetic aperture radars for space missions such as earth observation and studying evolution of black holes or other planets [10], [11]. Furthermore, the operation of large RF telescope arrays also require concurrence of individual telescopes' headings [12]. Finally, in smart camera networks, pointing consensus can be used for monitoring or surveillance purposes.

There have not been many works in the literature studying the pointing consensus as a cooperative control problem. In an earlier work, Zhang *et al.* [13] considered a concurrent targeting problem where all agents are positioned along a straight line, and there are two agents (leaders) with their heading vectors pointed already to the target. The decentralized control law in [13] is based on the geometric property of intersection angles and is able to guide all headings to match with the intersection of two leaders' heading vectors. The pointing consensus protocol in [14] relaxed the collinearity assumption on the agents' positions. However, the agents in [14] still need some pieces of *a priori* information on the common target, given as a desired heading vector for one leader agent and several subtended angles for the other agents. Thus, even in the two-dimensional (2-D) space, the pointing consensus problem has not been completely solved in [13] and [14]. As observed in [14], a main challenge of this problem is that the agents cannot consent their heading vectors without some knowledge on their (relative) positions in the space.

In this paper, we provide a decentralized solution to the weighted centroid pointing consensus problem in the 3-D space. We assume that each agent in the group has additional information on some bearing vectors toward its neighbors. This assumption is feasible for camera networks since the bearing vectors can be obtained from the camera. From the bearing vector measurements, the agents can estimate their positions in the network up to a translation and a scaling if the framework defined by

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the bearing measurement graph and their positions is *infinitesimally bearing rigid* (IBR) [15]. Next, for any given target in the space, by considering the target as a special agent, it will be shown that the target-and-n-agent framework is also IBR. Thus, although the agents do not know their exact positions, if they can obtain the estimated positions of itself and the target up to a translation and a scaling, they can find the exact heading vectors toward the target and control their heading vectors correspondingly. Based on this argument, we propose a group's centroid pointing consensus strategy by solving three smaller problems, namely, bearing-based network localization, target decision, and heading coordination. For each subproblem, we propose a corresponding control law and show that the combination of these control laws asymptotically directs all agents' headings toward the group centroid from almost all initial conditions. By employing this strategy, we achieve solutions of both the bearing-based network localization and the concurrent targeting problems. As formation control and network localization are dual problems, it is worth mentioning that formation control with point tracking objectives has been studied [16]–[18]. However, the work [16] focused on forming a formation around a source in 2-D space and the communication between the agents is restricted to be a ring graph. In [17], the agents can achieve a target formation and track the formation's centroid simultaneously. However, the work [17] assumed that the interagent distance measurements are available and thus it is different from the bearing-based setup in this paper. Also, the work [18] considered the task of building a formation around a target point using only bearing measurements. However, the control law in [18] only guarantees the target formation to be locally achieved.

Next, we provide further discussions on the set of common targets. It is shown that the set of constraints to determine the common target should be invariant with respect to a translation and a scaling of the whole framework and is corresponding to the set of weighted centroids of the positions of the agents. By this argument, we modified our pointing consensus algorithm so that the agents can target any weighted centroid of their positions. Furthermore, we formulate a decentralized version of the Fermat–Weber location problem (FWLP) [19] based on our pointing consensus framework. The FWLP asks to find the point that minimizes a weighted distance sum to a set of n noncollocated points in the space. As an important problem in operations research, the FWLP was extensively studied in a centralized manner [19]–[21]. Differently from existing works in the literature, we propose two decentralized solutions for solving the FWLP based on a combination of the bearing-based localization and finite-time consensus algorithms [22]. The two proposed solutions are inspired from the Weiszfeld algorithm and the gradient-descent algorithm, respectively [23], [24]. Assuming the bearing-based network localization dynamics has been at a steady state, in both proposed algorithms, the agents run some finite-time consensus dynamics in a given time span to calculate some auxiliary variables for updating an estimation of the Fermat–Weber point. After updating the estimation, each agent reinitializes the consensus dynamics with regard to the new estimate of the Fermat–Weber point. Iterating these processes, all agents asymptotically find the directions to the precise solution of the FWLP. We note that in different discrete-time formulations, this type of iterative algorithm has been studied, for example, in [4], [25]–[27]. A hybrid updating strategy for distributed observers has also been proposed in [28] for linear systems. However, the approach and convergence result in [28] are based on properties of linear systems. In contrast, the proposed algorithms in this paper hinge on finite-time stability theory [29].

We summarize the main theoretical contributions of this paper as follows. The first contribution is a strategy to solve the pointing consensus problem for any weighted centroid of n agents' positions. In solving the pointing consensus problem, an estimation law for the bearing-based network localization problem is proposed, the connection between bearing rigidity theory and the pointing consensus problem is exploited, and the invariant property of the constraints imposed on the common target is also discussed. The second contribution is a decentralized formulation and two bearing-based solutions of the FWLP. As far as we know, decentralized solutions of the FWLP have not yet been studied in the literature.

The rest of this paper is organized as follows. In Section II, we formulate the problem and recall some background on bearing rigidity theory. Next, in Section III, we propose a strategy for the pointing consensus problem and study the system under the proposed strategy. Then, we formulate and provide two decentralized solutions to the FWLP in Section IV. Section V contains the simulation results, and Section VI concludes the paper.

Notations: The d-dimensional space is denoted by $\mathbb{R}^d$. Let $\boldsymbol{y} = [y_1, \ldots, y_d]^\top$ be a vector in $\mathbb{R}^d$. We denote $|\boldsymbol{y}|^\alpha = [|y|_1^\alpha, \ldots, |y|_d^\alpha]^\top$, $\mathbf{sig}(\boldsymbol{y})^\alpha = [\mathrm{sgn}(y_1)|y_1|^\alpha, \ldots, \mathrm{sgn}(y_d)|y_d|^\alpha]^\top$, and $\mathbf{sgn}(\boldsymbol{y}) = [\mathrm{sgn}(y_1), \ldots, \mathrm{sgn}(y_d)]^\top$. The $n \times n$ identity matrix is denoted by $\boldsymbol{I}_n$. The $n \times 1$ vector of all ones is denoted by $\mathbf{1}_n$. For a matrix $\boldsymbol{A}$, we use $\mathcal{N}(\boldsymbol{A})$, $\mathcal{R}(\boldsymbol{A})$, and $r(A)$ to denote the nullspace, column space, and rank of $\boldsymbol{A}$, respectively. The orthogonal projection matrix corresponding to a nonzero vector $\boldsymbol{x} \in \mathbb{R}^3$ is defined as $\boldsymbol{P_x} \triangleq \boldsymbol{I}_3 - \boldsymbol{x}\boldsymbol{x}^\top/\|\boldsymbol{x}\|^2$. The matrix $\boldsymbol{P_x} \in \mathbb{R}^{3\times 3}$ is symmetric, positive semidefinite, and idempotent ($\boldsymbol{P_x} = \boldsymbol{P_x}^\top = \boldsymbol{P_x}^2 \geq 0$). The nullspace of $\boldsymbol{P_x}$ is spanned by $\boldsymbol{x}$, or, i.e., $\mathcal{N}(\boldsymbol{P_x}) = \mathcal{R}(\boldsymbol{x})$.

II. Problem Formulation

Consider an n-agent system ($n \geq 4$) in a 3-D ambient space. The location of each agent is unknown to itself and other agents. However, these agents have information about a common global reference frame. Information about the common global reference frame can be obtained distributedly by, for example, employing an orientation estimation strategy as in [30] and [31]. Let $\boldsymbol{p}_i \in \mathbb{R}^3, i \in \mathcal{I} \triangleq \{1, \ldots, n\}$, be the fixed position vector of the ith agent ($\dot{\boldsymbol{p}}_i(t) = \mathbf{0}$).[1]

[1] The assumption on stationary agents is reasonable to model a camera network. For a group of agents moving with the same linear velocity, i.e., $\dot{\boldsymbol{p}}_i = \boldsymbol{v}, \forall i \in \mathcal{I}$, the analysis will not be different after making changes of variables with regard to the common velocity, e.g., $\tilde{\boldsymbol{p}}_i = \boldsymbol{p}_i - \int_0^t \boldsymbol{v} d\tau$ and study $\tilde{\boldsymbol{p}}_i$ instead of $\boldsymbol{p}_i$ [18].

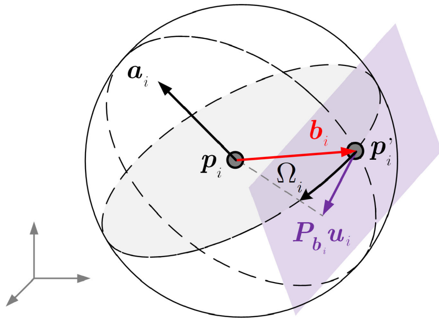

Fig. 1. Agent i does not know its position $\boldsymbol{p}_i$ but can reference its heading vector $\boldsymbol{b}_i$ in a global coordinate frame. The rotational motion of agent i's heading vector along axis $\boldsymbol{a}_i$ with rotational speed Ω_i can be equivalently represented as the linear velocity $\dot{\boldsymbol{p}}'_i = \boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{u}_i$ of the head point $\boldsymbol{p}'_i$.

Let each agent i have a heading direction given by a unit vector $\boldsymbol{b}_i \in \mathbb{R}^3$, $\|\boldsymbol{b}_i\| = 1$, as depicted in Fig. 1. Suppose that the agent can fully control its heading direction by rotating the heading around the point $\boldsymbol{p}_i$. Defining $\boldsymbol{p}'_i = \boldsymbol{p}_i + \boldsymbol{b}_i$, the rotational motion of $\boldsymbol{b}_i$ is equivalent to the motion of the point $\boldsymbol{p}'_i$ around the sphere of length 1 centered at $\boldsymbol{p}_i$.

The orthogonal projection matrix corresponding to $\boldsymbol{b}_i$ is given as $\boldsymbol{P}_{\boldsymbol{b}_i} = \boldsymbol{I}_3 - \boldsymbol{b}_i\boldsymbol{b}_i^\top$. Using the projection matrix, we can write the dynamics of the heading direction as

$$\dot{\boldsymbol{p}}'_i = \boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{u}_i \tag{1}$$

where $\boldsymbol{u}_i \in \mathbb{R}^3$ is the control input to be designed. Then

$$\dot{\boldsymbol{b}}_i = \frac{d}{dt}\left(\frac{\boldsymbol{p}'_i - \boldsymbol{p}_i}{\|\boldsymbol{p}'_i - \boldsymbol{p}_i\|}\right) = \boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{u}_i \tag{2}$$

where we have used the fact that $\dot{\boldsymbol{p}}_i = \boldsymbol{0}$ in (2). Note that the control input $\boldsymbol{u}_i$ to change $\boldsymbol{b}_i$ is introduced for design and analysis purposes. The rotational motion of $\boldsymbol{b}_i$ can be equivalently found from (2) as follows: $\dot{\boldsymbol{b}}_i = \boldsymbol{a}_i \wedge \boldsymbol{b}_i$, where $\boldsymbol{a}_i = \boldsymbol{b}_i \wedge \boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{u}_i$, and "$\wedge$" denotes the cross product. The vector $\boldsymbol{a}_i$ specifies the rotation plane and the angular velocity $\|\boldsymbol{a}_i\| = \|\boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{u}_i\| = |\Omega_i|\|\boldsymbol{b}_i\| = |\Omega_i|$, as shown in Fig. 1. Also, dynamics (2) can be considered as a single-integrator model subjected to a motion constraint [32].

In many applications, we would like all agents' headings to target a common point in space. The common point may be a target object[10], the centroid of all agents, or a location that minimizes a logistic function. The pointing consensus problem without position information has been shown to be a hard problem [13], [14]. In this paper, to remedy the lack of position information, we assume that each agent can measure the directional information (or the bearing vectors) with regard to a few neighboring agents. We will now formulate a pointing consensus problem, in which all agents' headings are desired to target the weighted average of the agents' positions.

A fixed, undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ characterizes the bearing sensing and information exchange graph between n agents, with the vertex set $\mathcal{V} = \{v_i | i \in \mathcal{I}\}$ of $|\mathcal{V}| = n$ vertices and the edge set $\mathcal{E} = \{e_{ij} = (v_i, v_j) | i, j \in \mathcal{I} \times \mathcal{I}, i \neq j\}$ of $|\mathcal{E}| = m$ edges. Consider an arbitrary indexing of all edges $\mathcal{E} = \{e_1, \ldots, e_m\}$, we use the following equivalent notations for the same edge $e_{k_{ij}} \equiv e_k \equiv e_{ij}$. For a given orientation of the edges, the incidence matrix $\boldsymbol{H} = [H_{ki}] \in \mathbb{R}^{m\times n}$ characterizes the relationship between the vertices and the edges in $\mathcal{G}$ and is defined such that $H_{ki} = -1$ if the edge $e_k \in \mathcal{E}$ leaves v_i, 1 if it enters vertex v_i, and 0 otherwise [33].

If there is an edge $e_{ij} \in \mathcal{E}$, two agents i and j can sense the bearing vector and exchange information (i.e., with communication) with regard to each other. The bearing vector from agent i to agent j is defined as

$$\boldsymbol{g}_{ij} \triangleq \frac{\boldsymbol{p}_j - \boldsymbol{p}_i}{\|\boldsymbol{p}_j - \boldsymbol{p}_i\|} = \frac{\boldsymbol{z}_{ij}}{\|\boldsymbol{z}_{ij}\|} \tag{3}$$

where $\boldsymbol{z}_{ij} \triangleq \boldsymbol{p}_j - \boldsymbol{p}_i$ is the displacement vector. It is easy to see that $\boldsymbol{g}_{ij} \in \mathbb{R}^3$ is a unit vector. Obviously, to define the bearing vector $\boldsymbol{g}_{ij}$, we require that two agents i and j are not collocated, i.e., $\boldsymbol{p}_i \neq \boldsymbol{p}_j, \forall i, j \in \mathcal{I}$. Let $\boldsymbol{p} = [\boldsymbol{p}_1^\top, \ldots, \boldsymbol{p}_n^\top]^\top \in \mathbb{R}^{3n}$ be the stacked vector of all agents' position vectors. We call $\boldsymbol{p}$ a configuration of the graph $\mathcal{G}$, and $\mathcal{G}(\boldsymbol{p})$ a framework in $\mathbb{R}^3$. Let $\boldsymbol{g} = [\ldots, \boldsymbol{g}_{k_{ij}}^\top, \ldots]^\top = [\boldsymbol{g}_1^\top, \ldots, \boldsymbol{g}_m^\top]^\top \in \mathbb{R}^{3m}$ be the stacked bearing vector. The bearing rigidity matrix is defined by [15]

$$\boldsymbol{R}(\boldsymbol{p}) \triangleq \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{p}} = \text{diag}\left(\frac{\boldsymbol{P}_{\boldsymbol{g}_k}}{\|\boldsymbol{z}_k\|}\right)(\boldsymbol{H} \otimes \boldsymbol{I}_3) \in \mathbb{R}^{3m\times 3n}. \tag{4}$$

We assume that the framework $\mathcal{G}(\boldsymbol{p})$ is IBR [15], that is, the rank of the bearing rigidity matrix is $r(\boldsymbol{R}(\boldsymbol{p})) = 3n - 4$. Intuitively, an IBR framework can be uniquely determined up to a translation and a scale factor from a set of bearing vectors $\{\boldsymbol{g}_{ij}\}_{(i,j)\in\mathcal{E}}$. For an IBR framework, the nullspace of $\boldsymbol{R}(\boldsymbol{p})$ is

$$\mathcal{N}(\boldsymbol{R}(\boldsymbol{p})) = \mathcal{R}([\mathbf{1}_n \otimes \boldsymbol{I}_3, \boldsymbol{p}]) = \mathcal{R}([\mathbf{1}_n \otimes \boldsymbol{I}_3, \boldsymbol{p} - \mathbf{1}_n \otimes \boldsymbol{p}_c])$$

where $\boldsymbol{p}_c \triangleq \sum_{i=1}^n \boldsymbol{p}_i/n$ is the group's centroid.

Let the positions of the agents span $\mathbb{R}^3$, i.e., $\mathcal{R}([\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n]) = \mathbb{R}^3$. The convex hull [34], [35] of a set of n points $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$ contains all points satisfying

$$\mathcal{S} = \left\{\sum_{i=1}^n \zeta_i \boldsymbol{p}_i \,\middle|\, \zeta_i \geq 0, \text{ and } \sum_{i=1}^n \zeta_i = 1\right\}.$$

We will refer to a point in $\mathcal{S}$ such that $\zeta_i > 0, \forall i \in \mathcal{I}$, as a *weighted centroid* of $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$ [17]. The positive number $\zeta_i > 0$ can be interpreted as the weight of agent's i opinion in deciding the common target of the pointing consensus problem.

Before stating the main problem, we list all main assumptions as follows.

Assumption 2.1: The agents have knowledge about a global reference frame. The time clocks of the agents are synchronized.[2] Each agent can control its heading vector according to (2).

Assumption 2.2: The communication graph $\mathcal{G}$ of n agents is fixed and undirected. The agents exchange their estimate variables over the graph $\mathcal{G}$. The framework $\mathcal{G}(\boldsymbol{p})$ is IBR.

[2]Note that it is important for the agents' clocks to be synchronized. Although this assumption is often preassumed in the literature, we emphasize this assumption since timing is important in both pointing consensus and our later proposed algorithms to the FWLP.

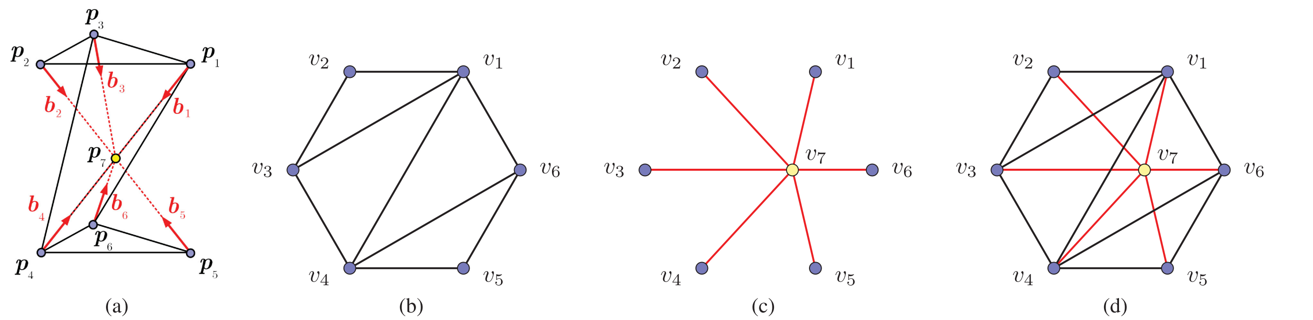

Fig. 2. Example: A six-agent system and a common target. (a) Agents consent their heading directions into a common point $\boldsymbol{p}_7$. The communication links between six agents are denoted by black lines. (b) Information graph $\mathcal{G}$. (c) Pointing graph (objective graph). (d) Union graph $\bar{\mathcal{G}}$.

The following section focuses on studying the following problem.

Problem 2.1: Given an n-agent system embedded in a 3-D ambient space satisfying Assumptions 2.1 and 2.2, design a decentralized control law for each agent using the bearing measurements such that all agents' headings asymptotically target a weighted centroid of $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$.

III. Decentralized Strategy for Weighted Centroid Pointing Consensus

In this section, we study the Problem 2.1. We first show that if the n-agent framework is IBR, then so is the union framework of n agents and the target. Second, we propose a control strategy comprised of three parts: 1) a bearing-based position estimation, 2) target determination, and 3) heading coordination to direct all the agents' headings toward the group's centroid $\boldsymbol{p}_c$, which is a specific weighted centroid of $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$. We provide a mathematical analysis to support the effectiveness of our control strategy. It will be proven that the proposed strategy almost globally asymptotically solves Problem 2.1 when the common target is the group centroid. Finally, we provide further analysis and show that the proposed centroid pointing consensus strategy can be modified to solve the weighted centroid pointing consensus problem in the 3-D space, or, i.e., Problem 2.1.

A. Bearing Rigidity and the Pointing Consensus Problem

Consider the n-agent system ($n \geq 4$) with a corresponding framework $\mathcal{G}(\boldsymbol{p})$ embedded in $\mathbb{R}^3$. Let $\boldsymbol{p}_{n+1}$ be the desired point that all agents' headings should point toward. Define the pointing graph $\mathcal{P}$ with the vertex set $\bar{\mathcal{V}} = \mathcal{V} \cup \{v_{n+1}\}$ and the edge set $\mathcal{E}(\mathcal{P}) = \{(v_i, v_{n+1}) | i \in \mathcal{I}\}$. The pointing graph $\mathcal{P}$ describes the group's objective, that is, each edge (v_i, v_{n+1}) implies that agent i needs to point toward the target point $\boldsymbol{p}_{n+1}$. Further, we define the graph $\bar{\mathcal{G}} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$, where $\bar{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}(\mathcal{P}) = \mathcal{E} \cup \{(v_i, v_{n+1}) | i \in \mathcal{I}\}$ as depicted in Fig. 2. Also, let $\bar{\boldsymbol{p}} = [\boldsymbol{p}^\top, \boldsymbol{p}_{n+1}^\top]^\top$. Observe that $\bar{\mathcal{G}}$ is a union graph of the bearing measurement graph $\mathcal{G}$ and the pointing graph. Since the agents have access to only the relative bearing vectors $\{\boldsymbol{g}_{ij}\}_{e_{ij} \in \mathcal{E}} \equiv \{\boldsymbol{g}_k\}_{k=1,\ldots,m}$, we can at best estimate the agents' positions up to translations and scales. We have the following result on the union framework $\bar{\mathcal{G}}(\bar{\boldsymbol{p}})$.

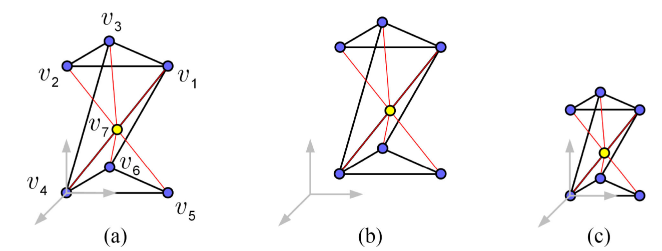

Fig. 3. Example: A six-agent system and a common target. (a) Configuration of the union framework $\bar{\mathcal{G}}(\bar{\boldsymbol{p}})$. The configuration $\bar{\boldsymbol{p}}$ after (b) a translation and (c) a dilation. For both cases (b) and (c), the agents' headings still target a point (yellow).

Lemma 3.1: Suppose that $\mathcal{G}(\boldsymbol{p})$ is IBR. Then, the union framework $\bar{\mathcal{G}}(\bar{\boldsymbol{p}})$ is IBR.

Proof: Note that we can treat the heading vectors $\{\boldsymbol{b}_k\}_{k=1,\ldots,n}$ and the bearing vectors $\{\boldsymbol{g}_k\}_{k=1,\ldots,m}$ in the union framework $\bar{\mathcal{G}}(\bar{\boldsymbol{p}})$ similarly since they are both unit vectors. The construction of $\bar{\mathcal{G}}$ can be decomposed into two steps. The first step is constructing $\mathcal{G}_1$ by a *Henneberg vertex addition* operation, that is, by adding the vertex v_{n+1} to the graph $\mathcal{G}$, together with edges connecting it to two previously existing vertices $v_i, v_j \in \mathcal{V}, i \neq j$ so that $\boldsymbol{p}_i, \boldsymbol{p}_j$, and $\boldsymbol{p}_{n+1}$ are not collinear.[3] It was shown in [36] that such an operation preserves the IBR property. The second step adds $n-2$ edges (v_k, v_{n+1}), $k \in \mathcal{I} \setminus \{i, j\}$ to $\mathcal{G}_1$ to generate $\bar{\mathcal{G}}$. Then, it follows from [36, Th. 2 and Lemma 6] that the framework $\bar{\mathcal{G}}(\bar{\boldsymbol{p}})$ is IBR. ■

Intuitively, Lemma 3.1 shows that by adding to an IBR graph a vertex that is fully connected to all the original vertices does not change the IBR property. Based on this lemma, if all agents' heading vectors are pointing to a common point, their heading vectors maintain pointing toward another common point when the union framework $\bar{\mathcal{G}}(\bar{\boldsymbol{p}})$ is translated or scaled. This argument is illustrated in Fig. 3. Thus, if the agents can somehow estimate their positions up to a translation and a scaling from the bearing measurements, they can control their heading vectors toward a common point determined by some distributed

[3]We can always find v_i, v_j to satisfy this condition because otherwise $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ are all collinear and thus $\mathcal{G}(\boldsymbol{p})$ cannot be IBR.

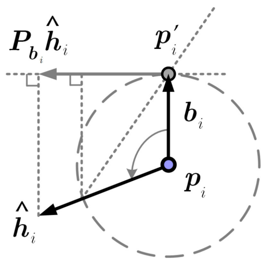

Fig. 4. Suppose that the vector $\hat{\boldsymbol{h}}_i$ is fixed, the control law (7) rotates $\boldsymbol{b}_i$ to align with $\hat{\boldsymbol{h}}_i$ exponentially fast.

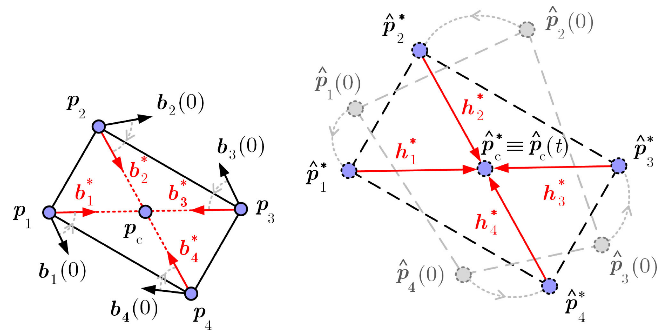

Fig. 5. Example 3.1: A four-agent system under the strategy (5)–(7). The true configuration $\boldsymbol{p} = [\boldsymbol{p}_1^\top, \ldots, \boldsymbol{p}_4^\top]^\top$ is different from the final estimated configuration $\hat{\boldsymbol{p}}^* = [\hat{\boldsymbol{p}}_1^{*\top}, \ldots, \hat{\boldsymbol{p}}_4^{*\top}]^\top$ only in a translation and a scaling factor.

protocols between them, and the pointing consensus problem is solved.

Note that the bearing-based network localization problem has been studied in [37] and [38]. In [38], Zhao and Zelazo assumed that there are several beacon nodes that have access to their absolute locations and proposed a network localization algorithm. Due to the existence of the beacon nodes, all other nodes can estimate their precise locations under the proposed algorithm in [38]. In the setup of this paper, since we assume no beacon node, the agents can only estimate their positions up to a translation and a scaling. It will be shown later that the agents do not need their absolute positions to solve Problem 2.1.

Remark 3.1: In many existing works in the literature, rigidity is important to the network localization or formation control task and is often studied separately from other objectives. For examples, the ideas of adding vertices and edges to an existing rigid framework to build a larger rigid one for studying formation control/network localization were presented in [33] and [36]. The result in this section shows that rigidity is important to both network localization and pointing consensus objectives. Further, the two objectives can be combined and considered simultaneously.

B. Centroid Pointing Consensus Strategy

Let agent $i \in \mathcal{I}$ in the system maintain an estimation of its position in $\hat{\boldsymbol{p}}_i \in \mathbb{R}^3$. By communicating through the information graph $\mathcal{G}$, the agents exchange the current estimations with their neighbors. Based on the exchanged estimations and bearing measurements, the agent i updates its position estimation under the following bearing-based estimation dynamics:

$$\dot{\hat{\boldsymbol{p}}}_i(t) = -\sum_{j\in\mathcal{N}_i} \boldsymbol{P}_{\hat{\boldsymbol{g}}_{ij}} \boldsymbol{g}_{ij} - \sum_{j\in\mathcal{N}_i} \|\hat{\boldsymbol{g}}_{ij} - \boldsymbol{g}_{ij}\| \boldsymbol{P}_{\hat{\boldsymbol{g}}_{ij}} (\mathbf{sgn}(\boldsymbol{P}_{\hat{\boldsymbol{g}}_{ij}} \boldsymbol{g}_{ij}) + \boldsymbol{n}_{ij}). \quad (5)$$

Note that in (5), $\hat{\boldsymbol{z}}_{ij} = \hat{\boldsymbol{p}}_j - \hat{\boldsymbol{p}}_i$, $\hat{\boldsymbol{g}}_{ij} = \hat{\boldsymbol{z}}_{ij}/\|\hat{\boldsymbol{z}}_{ij}\|$, and $\boldsymbol{P}_{\hat{\boldsymbol{g}}_{ij}} = \boldsymbol{I}_3 - \hat{\boldsymbol{g}}_{ij}\hat{\boldsymbol{g}}_{ij}^\top$ can be calculated by agent i from its estimation and its neighbors' estimations, while $\boldsymbol{g}_{ij}$ is measured from agent i. The perturbation term $\boldsymbol{n}_{ij}(t) = [n_{ij1}, n_{ij2}, n_{ij3}]^\top$ is a continuous time-varying vector satisfying $\|\boldsymbol{n}_{ij}(t)\| = \rho < 1$, and $\boldsymbol{n}_{ij} = -\boldsymbol{n}_{ji}, \forall e_{ij} \in \mathcal{E}$.

The bearing-based estimation law (5) consists of two parts: the first part $-\sum_{j\in\mathcal{N}_i} \boldsymbol{P}_{\hat{\boldsymbol{g}}_{ij}} \boldsymbol{g}_{ij}$ is the dual control law of the formation control law introduced in [15], and the remaining part is an adjustment term introduced to guarantee a global convergence of the estimation to the desired value. Note that if there is no error between the sensed and estimated bearing vectors $\|\hat{\boldsymbol{g}}_{ij} - \boldsymbol{g}_{ij}\| = 0, \forall j \in \mathcal{N}_i$, this term vanishes.

Remark 3.2: The adjustment term $\boldsymbol{n}_{ij}(t)$ has been introduced and discussed in formation control problems [39]–[41]. The system $\dot{\hat{\boldsymbol{p}}}_i = -\sum_{j\in\mathcal{N}_i} \boldsymbol{P}_{\hat{\boldsymbol{g}}_{ij}} \boldsymbol{g}_{ij}$ has an undesired equilibrium point, which is unstable [15]. The term $\boldsymbol{n}_{ij}(t)$ acts as a perturbation to drive the system out of this undesired equilibrium. In this paper, we set $\boldsymbol{n}_{ij}(t) = \rho_{ij}[\cos(\sigma_{ij}t), \sin(\sigma_{ij}t)\cos(\sigma t), \sin(\sigma_{ij}t)\sin(\sigma t)]^\top$. The parameters $\{\rho_{ij}, \sigma_{ij}\}_{j\in\mathcal{N}_i}$, and σ are given to agent i. By selecting the parameters such that $\rho_{ij} = -\rho_{ji}$, $0 < |\rho_{ij}| = \rho < 1$, $\forall (i,j) \in \mathcal{E}$, it is not difficult to check that $\|\boldsymbol{n}_{ij}(t)\|_2 = \rho < 1$, and $\boldsymbol{n}_{ij} = -\boldsymbol{n}_{ji}, \forall e_{ij} \in \mathcal{E}$.

Depending on application, the n agents may choose to consent their heading vectors toward a specific point in space. In this section, since we want all agents to point toward the centroid of the n agents, the following decentralized centroid estimation and pointing consensus dynamics for each agent $i \in \mathcal{I}$ are proposed:

$$\dot{\hat{\boldsymbol{q}}}_i(t) = \sum_{j\in\mathcal{N}_i} (\hat{\boldsymbol{q}}_j(t) - \hat{\boldsymbol{q}}_i(t)), \hat{\boldsymbol{q}}_i(0) = \hat{\boldsymbol{p}}_i(0) \quad (6)$$

$$\dot{\boldsymbol{b}}_i(t) = \boldsymbol{P}_{\boldsymbol{b}_i}(\hat{\boldsymbol{q}}_i(t) - \hat{\boldsymbol{p}}_i(t)). \quad (7)$$

Dynamics (6) is simply a consensus protocol used to determine the centroid of n estimated points $\hat{\boldsymbol{p}}_i(0), i \in \mathcal{I}$, so that the variable $\hat{\boldsymbol{q}}_i(t)$ contains the estimation of $\hat{\boldsymbol{p}}_c$ (the estimated group's centroid) by agent i at time t. Meanwhile, the pointing dynamics (7) guides the heading vector $\boldsymbol{b}_i$ to the true centroid. In (7), $\hat{\boldsymbol{h}}_i(t) = \hat{\boldsymbol{q}}_i(t) - \hat{\boldsymbol{p}}_i(t)$ is an estimation of the displacement vector from agent i toward the group's centroid and is time varying. The control law (7) was taken from [14] and [41] and its concept is illustrated in Fig. 4. In summary, our proposed centroid pointing strategy consists of three control laws (5)–(7) running simultaneously.

Example 3.1: To explain the concept of the pointing consensus strategy (5)–(7), consider a four-agent system as depicted in Fig. 5. The true configuration is $\boldsymbol{p} = [\boldsymbol{p}_1^\top, \ldots, \boldsymbol{p}_4^\top]^\top$. Initially, agent i's heading vector is $\boldsymbol{b}_i(0)$, it makes a random estimation

$\hat{\boldsymbol{p}}_i(0)$ and initializes $\hat{\boldsymbol{q}}_i(0) = \hat{\boldsymbol{p}}_i(0)$. Under (5)–(7), $\hat{\boldsymbol{p}}_i(t) \rightarrow \hat{\boldsymbol{p}}_i^*$, $\hat{\boldsymbol{p}}_c(t) \equiv \hat{\boldsymbol{p}}_c^* = \sum_{i=1}^4 \hat{\boldsymbol{p}}_i^*/4$, $\hat{\boldsymbol{q}}_i(t) \rightarrow \hat{\boldsymbol{p}}_c^*$, and $\boldsymbol{b}_i(t) \rightarrow \boldsymbol{b}_i^*, \forall i = 1, \ldots, 4$. Let $\hat{\boldsymbol{p}}^* = [\hat{\boldsymbol{p}}_1^{*\top}, \ldots, \hat{\boldsymbol{p}}_4^{*\top}]^\top$ be the final estimated configuration, $\mathcal{G}(\boldsymbol{p})$ is bearing congruent [15] to $\mathcal{G}(\hat{\boldsymbol{p}}^*)$ (i.e., $\frac{\boldsymbol{p}_j - \boldsymbol{p}_i}{\|\boldsymbol{p}_j - \boldsymbol{p}_i\|} = \frac{\boldsymbol{p}_j^* - \boldsymbol{p}_i^*}{\|\boldsymbol{p}_j^* - \boldsymbol{p}_i^*\|}, \forall i \neq j$). Also, let $\bar{\boldsymbol{p}} = [\boldsymbol{p}_1^\top, \ldots, \boldsymbol{p}_4^\top, \boldsymbol{p}_c^\top]^\top$ and $\bar{\hat{\boldsymbol{p}}}^* = [\hat{\boldsymbol{p}}_1^{*\top}, \ldots, \hat{\boldsymbol{p}}_4^{*\top}, \hat{\boldsymbol{p}}_c^{*\top}]^\top$, $\bar{\mathcal{G}}(\bar{\boldsymbol{p}})$ is bearing congruent to $\bar{\mathcal{G}}(\bar{\hat{\boldsymbol{p}}}^*)$ (see Lemma 3.1). It follows that $\boldsymbol{b}_i^* = \frac{\boldsymbol{p}_c - \boldsymbol{p}_i}{\|\boldsymbol{p}_c - \boldsymbol{p}_i\|} = \frac{\boldsymbol{h}_i^*}{\|\boldsymbol{h}_i^*\|}$, where $\boldsymbol{h}_i^* = \hat{\boldsymbol{p}}_c^* - \hat{\boldsymbol{p}}_i^*$. Thus, the heading vectors $\boldsymbol{b}_i$ asymptotically point to the group's centroid $\boldsymbol{p}_c$.

C. Stability Analysis

In this section, we will show that the proposed strategy (5)–(7) asymptotically drives all agents' headings toward their centroid. Let each agent initialize a random position estimation $\hat{\boldsymbol{p}}_i(0)$. Without loss of generality, we can assume that these initial estimated values are all different. Let $\hat{\boldsymbol{p}} = [\hat{\boldsymbol{p}}_1^\top, \ldots, \hat{\boldsymbol{p}}_n^\top]^\top$, $\boldsymbol{g} = [\boldsymbol{g}_1^\top, \ldots, \boldsymbol{g}_m^\top]^\top$, $\boldsymbol{n} = [\boldsymbol{n}_1^\top, \ldots, \boldsymbol{n}_m^\top]^\top$ with $\boldsymbol{n}_k = [n_{k1}, n_{k2}, n_{k3}]^\top \in \mathbb{R}^3, k = 1, \ldots, m$, and $\bar{\boldsymbol{H}} = \boldsymbol{H} \otimes \boldsymbol{I}_3$, we can rewrite (5) in the following compact form:

$$\begin{aligned}\dot{\hat{\boldsymbol{p}}} &= \bar{\boldsymbol{H}}^\top \mathrm{diag}\left(\boldsymbol{P}_{\hat{\boldsymbol{g}}_k}\right)\boldsymbol{g} \\ &\quad + \bar{\boldsymbol{H}}^\top \mathrm{diag}\left(\boldsymbol{P}_{\hat{\boldsymbol{g}}_k}\|\hat{\boldsymbol{g}}_k - \boldsymbol{g}_k\|\right)(\mathbf{sgn}(\mathrm{diag}(\boldsymbol{P}_{\hat{\boldsymbol{g}}_k})\boldsymbol{g}) + \boldsymbol{n}) \\ &= \tilde{\boldsymbol{R}}(\hat{\boldsymbol{p}})^\top(\boldsymbol{g} + \mathrm{diag}(\|\hat{\boldsymbol{g}}_k - \boldsymbol{g}_k\|\boldsymbol{I}_3)(\mathbf{sgn}(\mathrm{diag}(\boldsymbol{P}_{\hat{\boldsymbol{g}}_k})\boldsymbol{g}) + \boldsymbol{n}))\end{aligned} \tag{8}$$

where $\tilde{\boldsymbol{R}}(\hat{\boldsymbol{p}}) \triangleq \mathrm{diag}(\|\hat{\boldsymbol{z}}_k\|\boldsymbol{I}_3)\boldsymbol{R}(\hat{\boldsymbol{p}})$. Defining $\hat{\boldsymbol{p}}_c \triangleq (\sum_{i=1}^n \hat{\boldsymbol{p}}_i)/n$ and $s(\hat{\boldsymbol{p}}) \triangleq \sqrt{\sum_{i=1}^n \|\hat{\boldsymbol{p}}_i - \hat{\boldsymbol{p}}_c\|/n}$ as the estimated group's centroid and scale, respectively, we have the following lemma.

Lemma 3.2: Given the initial estimations $\hat{\boldsymbol{p}}_i(0), \forall i \in \mathcal{I}$, under the estimation dynamics (5), the estimated group's centroid and scale are time invariant.

Proof: Note that $\mathcal{N}(\tilde{\boldsymbol{R}}(\hat{\boldsymbol{p}})) = \mathcal{N}(\boldsymbol{R}(\hat{\boldsymbol{p}}))$. From (8) and properties of the bearing rigidity matrix $\boldsymbol{R}(\hat{\boldsymbol{p}})$, we have $\dot{\hat{\boldsymbol{p}}} \perp \mathcal{R}([\mathbf{1}_n \otimes \boldsymbol{I}_3, \hat{\boldsymbol{p}} - \mathbf{1}_n \otimes \hat{\boldsymbol{p}}_c])$, where $\perp$ denotes orthogonality. Thus, by writing $\hat{\boldsymbol{p}}_c = (\mathbf{1}_n^\top \otimes \boldsymbol{I}_3)\hat{\boldsymbol{p}}/n$ and $s(\hat{\boldsymbol{p}}) = \|\hat{\boldsymbol{p}} - \mathbf{1}_n \otimes \hat{\boldsymbol{p}}_c\|/\sqrt{n}$, it follows that

$$\dot{\hat{\boldsymbol{p}}}_c = (\mathbf{1}_n^\top \otimes \boldsymbol{I}_3)\dot{\hat{\boldsymbol{p}}}/n = \mathbf{0}, \dot{s}(\hat{\boldsymbol{p}}) = \frac{(\hat{\boldsymbol{p}} - \mathbf{1}_n \otimes \hat{\boldsymbol{p}}_c)^\top \dot{\hat{\boldsymbol{p}}}}{\sqrt{n}\|\hat{\boldsymbol{p}} - \mathbf{1}_n \otimes \hat{\boldsymbol{p}}_c\|} = 0$$

or, i.e., $\hat{\boldsymbol{p}}_c$ and $\dot{s}(\hat{\boldsymbol{p}})$ are time invariant. ■

Several remarks can be made from Lemma 3.2. First, since the sum $\sum_{i=1}^n \hat{\boldsymbol{p}}_i(t)$ is time invariant under (5), we can initiate $\hat{\boldsymbol{q}}_i(0) = \hat{\boldsymbol{p}}_i(0)$ in (6) without worrying about dynamics (5). Second, the invariance of the group's centroid and scale gives a constraint on the number of equilibria of (5). Finally, the estimation values will not diverge when evolving under (5) because otherwise the group's centroid ($\hat{\boldsymbol{p}}_c$) and the scale $s(\hat{\boldsymbol{p}})$ (which is mathematically the mean of deviation of the estimations $\hat{\boldsymbol{p}}_i$, $\forall i \in \mathcal{I}$, with regard to the group's centroid) cannot be invariant.

Next, we study convergence of the estimation law (5). Since (5) is a nonsmooth control law, we consider the solution of (8) in the Filippov sense [42], [43]. For brevity, we denote $\boldsymbol{\eta} = \mathrm{diag}(\boldsymbol{P}_{\hat{\boldsymbol{g}}_k})\boldsymbol{g} = [\boldsymbol{\eta}_1^\top, \ldots, \boldsymbol{\eta}_m^\top]^\top$, where each $\boldsymbol{\eta}_k = [\eta_{k1}, \eta_{k2}, \eta_{k3}]^\top$ is a vector in $\mathbb{R}^3$. Then, for almost all time

$$\dot{\hat{\boldsymbol{p}}} \in \bar{\boldsymbol{H}}^\top \mathrm{diag}\left(\boldsymbol{P}_{\hat{\boldsymbol{g}}_k}\right)(\boldsymbol{\eta} + \mathrm{diag}(\|\hat{\boldsymbol{g}}_k - \boldsymbol{g}_k\|\boldsymbol{I}_3)(K[\mathbf{sgn}](\boldsymbol{\eta}) + \boldsymbol{n})) \tag{9}$$

where $K[\boldsymbol{f}](\boldsymbol{x})$ denotes the Filippov set-valued mapping of $\boldsymbol{f}(\boldsymbol{x})$, and "$\in$" denotes the differential inclusion.

Let $\hat{\boldsymbol{p}}^* \triangleq [\hat{\boldsymbol{p}}_1^{*\top}, \ldots, \hat{\boldsymbol{p}}_n^{*\top}]^\top \in \mathbb{R}^{3n}$ be the point satisfying: 1) centroid: $(\mathbf{1}_n^\top \otimes \boldsymbol{I}_3)\hat{\boldsymbol{p}}^*/n = \hat{\boldsymbol{p}}_c$, 2) scale: $s(\hat{\boldsymbol{p}}^*) = s(\hat{\boldsymbol{p}})$, and 3) at $\hat{\boldsymbol{p}}^*$, the bearing vectors are $\hat{\boldsymbol{g}}_{ij} = \boldsymbol{g}_{ij}, \forall e_{ij} \in \mathcal{E}$. It can be checked that $\hat{\boldsymbol{p}}^*$ exists and is an equilibrium of (8) [15]. Let $\hat{\boldsymbol{z}}^* = [\ldots, \boldsymbol{z}_{ij}^{*\top}, \ldots]^\top = [\boldsymbol{z}_1^{*\top}, \ldots, \boldsymbol{z}_m^{*\top}]^\top$, then $\hat{\boldsymbol{z}}^* = \mathrm{diag}(\|\hat{\boldsymbol{z}}_k^*\|\boldsymbol{I}_3)\boldsymbol{g}$. We have the following theorem.

Theorem 3.1: Suppose that Assumptions 2.1 and 2.2 are satisfied and $\hat{\boldsymbol{p}}_i(0) \neq \hat{\boldsymbol{p}}_j(0), \forall i \neq j$. Under the estimation law (5), $\hat{\boldsymbol{p}}^*$ is globally asymptotically stable.

Proof: Consider the Lyapunov function $V = \frac{1}{2}\|\hat{\boldsymbol{p}} - \hat{\boldsymbol{p}}^*\|^2$, which is positive definite, continuously differentiable, and radially unbounded. At each point $\hat{\boldsymbol{p}}$, we have $\partial V = (\hat{\boldsymbol{p}} - \hat{\boldsymbol{p}}^*)$. Then, $\dot{V}$ exists almost everywhere (a.e.) and $\dot{V} \in^{\text{a.e.}} \dot{\tilde{V}}$, where

$$\begin{aligned}\dot{\tilde{V}} &= \bigcap_{\boldsymbol{\xi} \in \partial V} \boldsymbol{\xi}^\top \dot{\hat{\boldsymbol{p}}} \\ &= (\hat{\boldsymbol{p}} - \hat{\boldsymbol{p}}^*)^\top \bar{\boldsymbol{H}}^\top \mathrm{diag}(\boldsymbol{P}_{\hat{\boldsymbol{g}}_k}) \\ &\quad \cdot \Big(\boldsymbol{\eta} + \mathrm{diag}(\|\hat{\boldsymbol{g}}_k - \boldsymbol{g}_k\|\boldsymbol{I}_3)(K[\mathbf{sgn}](\boldsymbol{\eta}) + \boldsymbol{n})\Big) \\ &= -\boldsymbol{\eta}^\top \mathrm{diag}(\|\hat{\boldsymbol{z}}_k^*\|\boldsymbol{I}_3) \\ &\quad \cdot \Big(\boldsymbol{\eta} + \mathrm{diag}(\|\hat{\boldsymbol{g}}_k - \boldsymbol{g}_k\|\boldsymbol{I}_3)(K[\mathbf{sgn}](\boldsymbol{\eta}) + \boldsymbol{n})\Big) \\ &\leq -\sum_{k=1}^m \|\hat{\boldsymbol{z}}_k^*\|\Big(\boldsymbol{\eta}_k^\top \boldsymbol{\eta}_k + \|\hat{\boldsymbol{g}}_k - \boldsymbol{g}_k\|(\boldsymbol{\eta}_k^\top K[\mathbf{sgn}](\boldsymbol{\eta}_k) \\ &\qquad - |\boldsymbol{\eta}_k^\top \boldsymbol{n}_k|)\Big).\end{aligned}$$

From property of the sgn function, we have $\boldsymbol{\eta}_k^\top K[\mathbf{sgn}](\boldsymbol{\eta}_k) = \sum_{l=1}^3 |\eta_{kl}| = \|\boldsymbol{\eta}_k\|_1$. Further, for $\boldsymbol{n}_k = [n_{k1}, n_{k2}, n_{k3}]^\top$, it holds $|n_{kl}| \leq \sqrt{\sum_{l=1}^3 n_{kl}^2} = \|\boldsymbol{n}_k\|, \forall l = 1, 2, 3$. Thus, $|\boldsymbol{\eta}_k^\top \boldsymbol{n}_k| \leq \sum_{l=1}^3 |\eta_{kl} n_{kl}| \leq \sum_{l=1}^3 |\eta_{kl}||n_{kl}| \leq \rho \sum_{l=1}^3 |\eta_{kl}| = \rho\|\boldsymbol{\eta}_k\|_1$. By combining these inequalities, it follows that

$$\begin{aligned}\dot{\tilde{V}} &\leq -\sum_{k=1}^m \|\hat{\boldsymbol{z}}_k^*\|\boldsymbol{\eta}_k^\top \boldsymbol{\eta}_k - \sum_{k=1}^m (1 - \rho)\|\hat{\boldsymbol{z}}_k^*\|\|\hat{\boldsymbol{g}}_k - \boldsymbol{g}_k\|\|\boldsymbol{\eta}\|_1 \\ &\leq -\sum_{k=1}^m \|\hat{\boldsymbol{z}}_k^*\|\boldsymbol{\eta}_k^\top \boldsymbol{\eta}_k \leq 0.\end{aligned} \tag{10}$$

Note that $\dot{\tilde{V}} = 0$ if and only if $\hat{\boldsymbol{g}}_k = \boldsymbol{g}_k, \forall k = 1, \ldots, m$, or $\hat{\boldsymbol{g}}_k = -\boldsymbol{g}_k, \forall k = 1, \ldots, m$. However, the configuration corresponding to $\hat{\boldsymbol{g}}_k = -\boldsymbol{g}_k, \forall k = 1, \ldots, m$, is not an equilibrium of (9) due to the adjustment term $\boldsymbol{n}$. Therefore, based on LaSalle's invariance principle for nonsmooth system [42], $\hat{\boldsymbol{p}}(t)$ globally asymptotically converges to $\hat{\boldsymbol{p}}^*$. ■

Theorem 3.2: Under Assumptions 2.1 and 2.2, $\hat{\boldsymbol{p}} = \hat{\boldsymbol{p}}^*$ is a locally exponentially stable equilibrium of (8).

Proof: From inequality (10), we can follow similar steps as in [15, Th. 11] to prove the claim. ■

Let $\hat{\boldsymbol{p}}_c^* \triangleq \sum_{i=1}^n \hat{\boldsymbol{p}}_i^*/n$, since the formation centroid is time invariant, it follows that $\hat{\boldsymbol{p}}_c^* \equiv \hat{\boldsymbol{p}}_c(t), \forall t \geq 0$. Considering the consensus protocol (6), the following result is canonical.

Theorem 3.3: Under Assumptions 2.1 and 2.2, $\hat{\boldsymbol{q}}_i(t)$ exponentially converges to $\hat{\boldsymbol{p}}_c^*, \forall i \in \mathcal{I}$.

Proof: Under Assumption 2.2, $\mathcal{G}(\boldsymbol{p})$ is IBR. This implies that $\mathcal{G}$ is connected. Thus, under the consensus protocol (6), $\hat{\boldsymbol{q}}_i(t)$ converges to $\sum_{i=1}^n \hat{\boldsymbol{q}}_i(0)/n = \sum_{i=1}^n \hat{\boldsymbol{p}}_i(0)/n = \hat{\boldsymbol{p}}_c(0) = \hat{\boldsymbol{p}}_c^*$ exponentially fast [1], [2]. ■

Consider the pointing dynamics (7). Let $\boldsymbol{h}_i^* \triangleq \hat{\boldsymbol{p}}_c^* - \hat{\boldsymbol{p}}_i^*$, we can rewrite (7) as follows:

$$\dot{\boldsymbol{b}}_i = \underbrace{\boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{h}_i^*}_{\triangleq \boldsymbol{f}_i(\boldsymbol{b}_i)} + \underbrace{\boldsymbol{P}_{\boldsymbol{b}_i}\left(-\boldsymbol{h}_i^* + \hat{\boldsymbol{h}}_i(t)\right)}_{\triangleq \boldsymbol{r}_i(t)}. \tag{11}$$

The heading control input (11) consists of two parts: the first part $[\boldsymbol{f}_i(\boldsymbol{b}_i)]$ depends only on $\boldsymbol{b}_i$ and the second part $[\boldsymbol{r}_i(t)]$ depends on the estimation dynamics (5)–(6). The following lemma states that the inputs only change the direction, and not the magnitude, of the heading vector $\boldsymbol{b}_i$.

Lemma 3.3: Under the control strategy (5)–(7), $\|\boldsymbol{b}_i(t)\| = 1$, $\forall i \in \mathcal{I}$, and $\forall t \geq 0$.

Proof: It can be verified that $\boldsymbol{b}_i^\top \dot{\boldsymbol{b}}_i = \boldsymbol{b}_i^\top \boldsymbol{P}_{\boldsymbol{b}_i}\hat{\boldsymbol{h}}_i = 0$ since $\boldsymbol{b}_i^\top \boldsymbol{P}_{\boldsymbol{b}_i} = \boldsymbol{0}_d^\top$. As a result, $\|\boldsymbol{b}_i(t)\| = \|\boldsymbol{b}_i(0)\| = 1, \forall t \geq 0$. ■

The following lemma is about the external input $\boldsymbol{r}_i(t)$.

Lemma 3.4: Suppose that Assumptions 2.1 and 2.2 hold. Then, the input $\boldsymbol{r}_i(t)$ in (11) is bounded and $\|\boldsymbol{r}_i(t)\| \to 0$ exponentially fast.

Proof: To show the boundedness property, we employ the following inequality $\|\boldsymbol{r}_i(t)\| = \|\boldsymbol{P}_{\boldsymbol{b}_i}(-\boldsymbol{h}_i^* + \hat{\boldsymbol{h}}_i)\| \leq \|\boldsymbol{P}_{\boldsymbol{b}_i}\|(\|\boldsymbol{h}_i^*\| + \|\hat{\boldsymbol{h}}_i\|)$. Note that $\|\boldsymbol{P}_{\boldsymbol{b}_i}\| = 1$, $\|\boldsymbol{h}_i^*\|$ is bounded, and $\|\hat{\boldsymbol{h}}_i\| \leq \|\hat{\boldsymbol{q}}_i\| + \|\hat{\boldsymbol{p}}_i\|$ is also bounded due to Theorems 3.1 and 3.3. Thus, $\boldsymbol{r}_i(t)$ is bounded. Moreover, it follows from Theorems 3.1 to 3.3 that

$$\hat{\boldsymbol{h}}_i(t) = \hat{\boldsymbol{q}}_i - \hat{\boldsymbol{p}}_i \to \hat{\boldsymbol{p}}_c^* - \hat{\boldsymbol{p}}_i^* = \boldsymbol{h}_i^*$$

exponentially as $t \to \infty$. Therefore, $\|\boldsymbol{r}_i(t)\| = \|\hat{\boldsymbol{h}}_i - \boldsymbol{h}_i^*\| \to 0$ as $t \to \infty$ and the convergence is exponentially fast. ■

Next, we consider the system

$$\dot{\boldsymbol{b}}_i = \boldsymbol{f}(\boldsymbol{b}_i) = \boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{h}_i^* \tag{12}$$

which is the system (11) without the input $\boldsymbol{r}(t)$. For $\boldsymbol{h}_i^* \neq 0$, we have the following lemma whose proof is similar to the proof of [14, Lemma 3.1] and will be omitted.

Lemma 3.5: System (12) has two equilibria $\boldsymbol{b}_i = \pm\boldsymbol{b}_i^*$, where $\boldsymbol{b}_i^* = \boldsymbol{h}_i^*/\|\boldsymbol{h}_i^*\|$. The equilibrium $\boldsymbol{b}_i = \boldsymbol{b}_i^*$ is almost globally exponentially stable and the equilibrium $\boldsymbol{b}_i = -\boldsymbol{b}_i^*$ is (exponentially) unstable.

We can now state the main result of this section.

Theorem 3.4: Suppose that Assumptions 2.1 and 2.2 hold. Under the control strategy (5)–(7), all agents' headings asymptotically point toward the group's centroid for almost all initial estimates $\hat{\boldsymbol{p}}(0)$.

Proof: Based on Lemma 3.4, for the heading vector $\boldsymbol{b}_i(t)$ to maintain at the undesired equilibrium $\boldsymbol{b}_i(t) \equiv -\boldsymbol{b}_i^*$, it is required that $\boldsymbol{r}_i(t) \equiv \boldsymbol{0}$ and $\boldsymbol{b}_i(0) = -\boldsymbol{b}_i^*$. Next, $\boldsymbol{r}_i(t) \equiv \boldsymbol{0}$ implies that $\dot{\hat{\boldsymbol{q}}}_i(t) \equiv \boldsymbol{0}$, $\dot{\hat{\boldsymbol{p}}}_i(t) \equiv \boldsymbol{0}$ and either $\hat{\boldsymbol{q}}_i(t) - \hat{\boldsymbol{p}}_i(t) \in \mathcal{R}(\boldsymbol{b}_i^*)$ or $\hat{\boldsymbol{q}}_i(t) = \hat{\boldsymbol{p}}_i(t)$. Notice that $\dot{\hat{\boldsymbol{q}}}_i(t) \equiv \boldsymbol{0}$ implies that $\hat{\boldsymbol{q}}_i(0) = \hat{\boldsymbol{p}}_c^*$, and $\dot{\hat{\boldsymbol{p}}}_i(t) \equiv \boldsymbol{0}$ implies $\hat{\boldsymbol{p}}_i(0) = \hat{\boldsymbol{p}}_i^*$. Thus, there are two cases: (i) $\hat{\boldsymbol{q}}_i(0) = \hat{\boldsymbol{p}}_c^*$ and $\hat{\boldsymbol{p}}_i(0) = \hat{\boldsymbol{p}}_i^*$, or (ii) $\hat{\boldsymbol{q}}_i(0) = \hat{\boldsymbol{p}}_i^*$ and $\hat{\boldsymbol{p}}_i(0) = \hat{\boldsymbol{p}}_i^*$. Since we only consider $\hat{\boldsymbol{p}}_i(0) \neq \hat{\boldsymbol{p}}_j(0)$ for all $i, j \in \mathcal{I}$, $i \neq j$, both cases (i) and (ii) lead to contradictions and thus $\boldsymbol{b}_i$ will not stay at the undesired equilibrium $\boldsymbol{b}_i = -\boldsymbol{b}_i^*$.

Consider the Lyapunov function $V = \frac{1}{2}\|\boldsymbol{b}_i - \boldsymbol{b}_i^*\|^2$, which is positive definite and continuously differentiable. At any point $\boldsymbol{b}_i \in \mathbb{R}^3$, we have

$$\begin{aligned}
\dot{V} &= (\boldsymbol{b}_i - \boldsymbol{b}_i^*)^\top \boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{h}_i^* + (\boldsymbol{b}_i - \boldsymbol{b}_i^*)^\top \boldsymbol{P}_{\boldsymbol{b}_i}\left(\hat{\boldsymbol{h}}_i(t) - \boldsymbol{h}_i^*\right) \\
&= -\boldsymbol{b}_i^{*\top}\boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{h}_i^* - \boldsymbol{b}_i^{*\top}\boldsymbol{P}_{\boldsymbol{b}_i}\left(\hat{\boldsymbol{h}}_i(t) - \boldsymbol{h}_i^*\right) \\
&\leq -\|\boldsymbol{h}_i^*\|\boldsymbol{b}_i^{*\top}\boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{b}_i^* + \|\boldsymbol{b}_i^{*\top}\boldsymbol{P}_{\boldsymbol{b}_i}\left(\hat{\boldsymbol{h}}_i(t) - \boldsymbol{h}_i^*\right)\| \\
&\leq -\beta_i\boldsymbol{b}_i^{*\top}\boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{b}_i^* + \|\boldsymbol{b}_i^{*\top}\boldsymbol{P}_{\boldsymbol{b}_i}\|\|\hat{\boldsymbol{h}}_i(t) - \boldsymbol{h}_i^*\|
\end{aligned} \tag{13}$$

where $\beta_i = \|\boldsymbol{h}_i^*\| > 0$. It follows from Lemma 3.4 that there exist $\delta_i, \gamma_i > 0$ such that $\|\hat{\boldsymbol{h}}_i(t) - \boldsymbol{h}_i^*\| \leq \beta_i\delta_i e^{-\gamma_i t}$. Thus

$$\dot{V} \leq -\beta_i\|\boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{b}_i^*\|\left(\|\boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{b}_i^*\| - \delta_i e^{-\gamma_i t}\right) \leq \frac{1}{4}\beta_i\delta_i^2 e^{-2\gamma_i t}$$

where the inequality holds if and only if $\|\boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{b}_i^*\| = \frac{\delta_i}{2}e^{-\gamma_i t}$. Thus

$$V(\infty) - V(0) \leq \int_0^\infty \frac{1}{4}\beta_i\delta_i^2 e^{-2\gamma_i\tau}d\tau = \frac{\beta_i\delta_i^2}{2\gamma_i} \tag{14}$$

which shows that V is bounded. Consider the function

$$W = \int_0^t \boldsymbol{b}_i^{*\top}\boldsymbol{P}_{\boldsymbol{b}_i}\left(\hat{\boldsymbol{h}}_i(\tau) - \boldsymbol{h}_i^*\right)d\tau \tag{15}$$

which is bounded because

$$\begin{aligned}
\|W\| &\leq \left\|\int_0^t \boldsymbol{b}_i^{*\top}\boldsymbol{P}_{\boldsymbol{b}_i}\left(\hat{\boldsymbol{h}}_i(\tau) - \boldsymbol{h}_i^*\right)d\tau\right\| \\
&\leq \int_0^t \|\boldsymbol{b}_i^{*\top}\boldsymbol{P}_{\boldsymbol{b}_i}\left(\hat{\boldsymbol{h}}_i(\tau) - \boldsymbol{h}_i^*\right)\|d\tau \\
&\leq \int_0^t \|\boldsymbol{b}_i^{*\top}\boldsymbol{P}_{\boldsymbol{b}_i}\|\|\hat{\boldsymbol{h}}_i(\tau) - \boldsymbol{h}_i^*\|d\tau \\
&\leq \int_0^t \beta_i\delta_i e^{-\gamma_i\tau}d\tau \leq \frac{\beta_i\delta_i}{\gamma_i}(1 - e^{-\gamma_i t}).
\end{aligned} \tag{16}$$

Consider $U = V + W$, then U is lower bounded, and

$$\dot{U} = \dot{V} + \dot{W} = -\beta_i\|\boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{b}_i^*\|^2 \leq 0. \tag{17}$$

Since $\|\dot{\boldsymbol{b}}_i\| = \|\boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{b}_i^*\| \leq \|\boldsymbol{P}_{\boldsymbol{b}_i}\|\|\boldsymbol{b}_i^*\| = 1$, $\dot{\boldsymbol{b}}_i$ is bounded and thus so is $\ddot{U}$. By Barbalat's lemma, $\lim_{t\to\infty}\dot{U} = 0$. Thus, $\|\boldsymbol{P}_{\boldsymbol{b}_i}\boldsymbol{b}_i^*\| \to 0$, or, i.e., $\boldsymbol{b}_i \to \pm\boldsymbol{b}_i^*$ as $t \to \infty$. However, the system (7) could not stay at $\boldsymbol{b}_i = -\boldsymbol{b}_i^*$ according to the discussion at the beginning of the proof. Therefore, we conclude that $\boldsymbol{b}_i \to \boldsymbol{b}_i^*, \forall i \in \mathcal{I}$, as $t \to \infty$. ■

We have several remarks to conclude this section.

Remark 3.3: The bearing-based position estimation dynamics (5) may fail if at some time t, there exists $\hat{\boldsymbol{p}}_i(t) = \hat{\boldsymbol{p}}_j(t)$, for $(v_i, v_j) \in \mathcal{E}$, and thus $\hat{\boldsymbol{g}}_{ij}$ is undefined. In this situation, we can follow [15] to obtain a sufficient condition on the initial estimation of $\{\hat{\boldsymbol{p}}_i(0)\}_{i\in\mathcal{I}}$ so that $\hat{\boldsymbol{p}}_i(t) \neq \hat{\boldsymbol{p}}_j(t), \forall i,j \in \mathcal{I}$. Other possible resolutions include reinitializing dynamics (5) if failure happens, or running (5) with a few sets of different initial estimations to reduce the possibility of computation failure.

Remark 3.4: As the agents do not have information on their global positions, they cannot localize the correct position of $\boldsymbol{p}_c$. However, they can determine precisely the direction toward $\boldsymbol{p}_c$ and point toward it. If there are few agents having their global positions and acting as beacon nodes, all agents can estimate their true positions as well as the true position of $\boldsymbol{p}_c$.

Remark 3.5: From the proposed strategy, if we replace the heading dynamics (7) by

$$\dot{\boldsymbol{b}}_i(t) = \boldsymbol{P}_{\boldsymbol{b}_i}\hat{\boldsymbol{q}}_i(t) \tag{18}$$

all agents will asymptotically point to the same direction $\boldsymbol{p}_c/\|\boldsymbol{p}_c\|$ as $t \to \infty$. On the other hand, if we omit the target decision dynamics (6) and replace the heading dynamics (7) by

$$\dot{\boldsymbol{b}}_i(t) = \boldsymbol{P}_{\boldsymbol{b}_i}(\boldsymbol{0} - \hat{\boldsymbol{p}}_i(t)) = -\boldsymbol{P}_{\boldsymbol{b}_i}\hat{\boldsymbol{p}}_i(t) \tag{19}$$

then the control laws (5), (19) asymptotically drive all the agents' headings to a common point. In general, if there exists a leader agent who selects a virtual target point $\hat{\boldsymbol{p}}_t$ and sends this information to other agents, then under (5) and $\dot{\boldsymbol{b}}_i(t) = \boldsymbol{P}_{\boldsymbol{b}_i}(\hat{\boldsymbol{p}}_t - \hat{\boldsymbol{p}}_i(t))$, we have $\boldsymbol{h}_i^* = \hat{\boldsymbol{p}}_t - \hat{\boldsymbol{p}}_i^*$ and the agents' headings will target $\boldsymbol{p}_t$ satisfying $\frac{\boldsymbol{p}_t - \boldsymbol{p}_i}{\|\boldsymbol{p}_t - \boldsymbol{p}_i\|} = \frac{\hat{\boldsymbol{p}}_t - \boldsymbol{p}_i^*}{\|\hat{\boldsymbol{p}}_t - \boldsymbol{p}_i^*\|}, \forall i \in \mathcal{I}$.

D. Further Analysis on the Desired Target Point

In this section, we discuss the target point in the pointing consensus problem. In general, dynamics (6) can be replaced by a more general target decision dynamics, which estimates $\hat{\boldsymbol{p}}_{n+1}$—an estimation of the target point $\boldsymbol{p}_{n+1}$. The target point $\boldsymbol{p}_{n+1}$ satisfies a set of predefined constraints. To guarantee that the agents can consent their headings toward $\boldsymbol{p}_{n+1}$ asymptotically under our proposed pointing consensus strategy, the set of constraints cannot be arbitrarily chosen.

When the estimation dynamics (8) is at its equilibrium, the agents can estimate the configuration $\boldsymbol{p}$ up to a translation and a scaling. Thus, we can write

$$\boldsymbol{p} = k_s(\hat{\boldsymbol{p}}^* - \boldsymbol{\Delta} \otimes \boldsymbol{1}_n) \tag{20}$$

where $\hat{\boldsymbol{p}}^*$ is the desired equilibrium of the estimation dynamics (8), $0 \neq k_s \in \mathbb{R}$ denotes a scale factor, and $\boldsymbol{\Delta} \in \mathbb{R}^3$ is a translation vector. Moreover, as the union framework $\bar{\mathcal{G}}(\bar{\boldsymbol{p}})$ is IBR, it follows that

$$\bar{\boldsymbol{p}} = k_s(\bar{\hat{\boldsymbol{p}}}^* - \boldsymbol{\Delta} \otimes \boldsymbol{1}_{n+1}) \tag{21}$$

where $\bar{\hat{\boldsymbol{p}}}^* = [\hat{\boldsymbol{p}}_1^{*\top}, \ldots, \hat{\boldsymbol{p}}_n^{*\top}, \hat{\boldsymbol{p}}_{n+1}^{*\top}]^\top \in \mathbb{R}^{3(n+1)}$.

Let $\boldsymbol{f}(\bar{\boldsymbol{p}}) = \boldsymbol{f}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, \boldsymbol{p}_{n+1}) = \boldsymbol{0}$ be the set of constraints that the target point needs to satisfy and assume that the set of constraints is sufficient to solve for $\boldsymbol{p}_{n+1}$. Then, we have the following result.

Theorem 3.5: The agents can determine the directions toward the designed target if and only if the set of constraints $\boldsymbol{f}(\bar{\boldsymbol{p}}) = \boldsymbol{0}$ that the target point needs to hold is invariant with respect to a translation and a scaling of the whole framework $\bar{\mathcal{G}}(\bar{\boldsymbol{p}})$, or, i.e., the set of constraints satisfies (21).

Proof: (Necessity) Suppose that the constraints $\boldsymbol{f}(\bar{\boldsymbol{p}}) = \boldsymbol{0}$ are invariant with respect to a translation and a scaling of the whole framework. It follows that $\boldsymbol{f}(\bar{\boldsymbol{p}}) = \boldsymbol{f}(k_s(\bar{\hat{\boldsymbol{p}}}^* - \boldsymbol{\Delta} \otimes \boldsymbol{1}_{n+1})) = \boldsymbol{f}(\bar{\hat{\boldsymbol{p}}}^*) = \boldsymbol{0}$. Thus, the estimated target point $\hat{\boldsymbol{p}}_{n+1}^*$ satisfies $\boldsymbol{f}(\bar{\hat{\boldsymbol{p}}}^*) = \boldsymbol{0}$. This implies that the agents can determine $\hat{\boldsymbol{p}}_{n+1}^*$ from the estimated positions $\hat{\boldsymbol{p}}_i^*, \forall i \in \mathcal{I}$, and the constraint $\boldsymbol{f}(\bar{\hat{\boldsymbol{p}}}^*) = \boldsymbol{0}$.

(Sufficiency) Suppose that the agents can determine the directions toward the designed target. Since the agents can estimate their positions $\hat{\boldsymbol{p}}_i^*, i \in \mathcal{I}$, differently from their precise positions by a translation and a scale factor, they can estimate the target $\hat{\boldsymbol{p}}_{n+1}^*$ by solving $\boldsymbol{f}(\bar{\hat{\boldsymbol{p}}}^*) = \boldsymbol{f}(\hat{\boldsymbol{p}}_1^*, \ldots, \hat{\boldsymbol{p}}_n^*, \hat{\boldsymbol{p}}_{n+1}^*) = \boldsymbol{0}$. Suppose that $\boldsymbol{f}(\bar{\boldsymbol{p}}) = \boldsymbol{0}$ is not invariant with respect to a translation and a scaling of the framework, or, i.e., $\boldsymbol{f}(\bar{\hat{\boldsymbol{p}}}^*) = \boldsymbol{f}(\hat{\boldsymbol{p}}_1^*, \ldots, \hat{\boldsymbol{p}}_n^*, \hat{\boldsymbol{p}}_{n+1}^*) \neq \boldsymbol{0}$. Then, the agents cannot correctly determine $\hat{\boldsymbol{p}}_{n+1}^*$, which implies that they cannot point toward the designed target. This leads to a contradiction. Thus, $\boldsymbol{f}(\bar{\boldsymbol{p}}) = \boldsymbol{0}$ needs to hold with respect to a translation and a scaling of the whole framework to guarantee that the agents can determine precisely the directions toward the designed target. ■

Note that invariant properties are often present in problems in multiagent systems. Theorem 3.5 is about bearing rigidity (or parallel rigidity) preserving motions [44]. Other invariances in network systems can be found in the literature, for example, see [45] and [46].

We will discuss in detail two special classes of the target point to illustrate Theorem 3.5. First, considering the following linear constraint:

$$\boldsymbol{f}(\bar{\boldsymbol{p}}) = \sum_{i=1}^{n+1} a_i \boldsymbol{p}_i = \boldsymbol{0}, \text{and } \sum_{i=1}^{n+1} a_i = 0 \tag{22}$$

where $a_i \in \mathbb{R}, i = 1, \ldots, n+1$, we prove the following theorem.

Theorem 3.6: Constraint (22) is invariant with respect to a translation, a scaling, and a rotation of the whole framework.

Proof: Since (22) is linear, we can separately check the invariance of (22) with respect to each operator.

1) Translation: Let $\hat{\boldsymbol{p}}_i^* = \boldsymbol{p}_i + \boldsymbol{\Delta}, \forall i = 1, \ldots, n+1$. Then

$$\boldsymbol{f}(\bar{\boldsymbol{p}}^*) = \sum_{i=1}^{n+1} a_i \boldsymbol{p}_i^* = \sum_{i=1}^{n+1} a_i \boldsymbol{p}_i^* + \left(\sum_{i=1}^{n+1} a_i\right)\boldsymbol{\Delta} = \boldsymbol{0}.$$

2) Scaling: Let $\bar{\hat{\boldsymbol{p}}}^* = k_s(\bar{\boldsymbol{p}} - \boldsymbol{\Delta} \otimes \boldsymbol{1}_{n+1})$. It follows that

$$\begin{aligned}\boldsymbol{f}(\bar{\hat{\boldsymbol{p}}}^*) &= \sum_{i=1}^{n+1} a_i \hat{\boldsymbol{p}}_i^* \\ &= k_s \sum_{i=1}^{n+1} a_i \boldsymbol{p}_i + k_s \left(\sum_{i=1}^{n+1} a_i\right)\boldsymbol{\Delta} = \boldsymbol{0}.\end{aligned}$$

3) Rotation: Without loss of generality, consider the rotation about $\boldsymbol{p}_1$ by the rotation matrix $\boldsymbol{Q} \in SO(3)$. We have

$\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{p}}_1^* = \boldsymbol{Q}(\boldsymbol{p}_i - \boldsymbol{p}_1)$, $i = 1, ..., n+1$. Then

$$\begin{aligned}\boldsymbol{f}(\bar{\hat{\boldsymbol{p}}}^*) &= \sum_{i=1}^{n+1} a_i \hat{\boldsymbol{p}}_i^* = \sum_{i=1}^{n+1} a_i (\hat{\boldsymbol{p}}_1^* + \boldsymbol{Q}(\boldsymbol{p}_i - \boldsymbol{p}_1)) \\ &= \left(\sum_{i=1}^{n+1} a_i\right) \hat{\boldsymbol{p}}_1^* + \boldsymbol{Q}\boldsymbol{f}(\bar{\boldsymbol{p}}) - \boldsymbol{Q}\left(\sum_{i=1}^{n+1} a_i\right) \boldsymbol{p}_1 \\ &= \boldsymbol{0}.\end{aligned}$$

Thus, (22) is invariant with respect to a translation, a scaling, and a rotation of the whole framework. ■

Now, suppose further in (22) that $a_i > 0, \forall i \in \mathcal{I}$, and the position estimation dynamics (5) is at equilibrium $\hat{\boldsymbol{p}} = \hat{\boldsymbol{p}}^*$. Let the target decision dynamics (6) in our proposed pointing consensus strategy be replaced by

$$a_i \dot{\hat{\boldsymbol{q}}}_i(t) = \sum_{j \in \mathcal{N}_i} (\hat{\boldsymbol{q}}_j(t) - \hat{\boldsymbol{q}}_i(t)), \hat{\boldsymbol{q}}_i(0) = \hat{\boldsymbol{p}}_i^* \ \ \forall i \in \mathcal{I}. \quad (23)$$

From [1, Corollary 3], we have $\sum_{i=1}^n a_i \hat{\boldsymbol{q}}_i(t) = \sum_{i=1}^n a_i \hat{\boldsymbol{q}}_i(0) = \sum_{i=1}^n a_i \boldsymbol{p}_i^* = -a_{n+1} \hat{\boldsymbol{p}}_{n+1}^*, \forall t \geq 0$ and

$$\hat{\boldsymbol{q}}_i(t) \to \frac{\sum_{i=1}^n a_i \hat{\boldsymbol{q}}_i(0)}{\sum_{i=1}^n a_i} = \frac{-a_{n+1} \hat{\boldsymbol{p}}_{n+1}^*}{-a_{n+1}} = \hat{\boldsymbol{p}}_{n+1}^*$$

as $t \to \infty$. Thus, dynamics (23) can asymptotically determine the target satisfying constraint (22). On the other hand, consider constraint (22). It is easy to check that the target point satisfies $\hat{\boldsymbol{p}}_{n+1}^* = \sum_{i=1}^n \frac{a_i}{\sum_{i=1}^n a_i} \hat{\boldsymbol{p}}_n^*$. Let $\zeta_i \triangleq \frac{a_i}{\sum_{i=1}^n a_i}$, it is clear that $\zeta_i > 0$ and $\sum_{i=1}^n \zeta_i = \sum_{i=1}^n \frac{a_i}{\sum_{i=1}^n a_i} = 1$, which implies that $\hat{\boldsymbol{p}}_{n+1}^*$ is a weighted centroid of $\{\hat{\boldsymbol{p}}_1^*, \ldots, \hat{\boldsymbol{p}}_n^*\}$. We can thus state the following theorem whose proof is similar to Theorem 3.4 and will be omitted.

Theorem 3.7: Suppose that the Assumptions 2.1 and 2.2 hold and each agent i knows $a_i > 0$. Under the control strategy (5), (7), (23), all agents' headings asymptotically target the weighted centroid of $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$ given by $\boldsymbol{p}_{n+1} = \sum_{i=1}^n \frac{a_i}{\sum_{i=1}^n a_i} \boldsymbol{p}_n$.

Remark 3.6: In Theorem 3.7, if the position estimation dynamics (6) is not at equilibrium, the dynamic average consensus proposed in [47] or [48] can be used instead.

Next, we revisit the group's centroid in light of Theorem 3.6. By rewriting equation $\boldsymbol{p}_{n+1} \equiv \boldsymbol{p}_c = \frac{1}{n} \sum_{i=1}^n \boldsymbol{p}_i$ as

$$\frac{1}{n} \sum_{i=1}^n \boldsymbol{p}_i - 1 \cdot \boldsymbol{p}_{n+1} = \boldsymbol{0}$$

and denoting $a_i = 1/n, \forall i \in \mathcal{I}$, and $a_{n+1} = -1$, it follows that $\sum_{i=1}^{n+1} a_i = 0$. Thus, the centroid's equation belongs to the class of constraints (22).

Remark 3.7: The class of constraints

$$\boldsymbol{f}(\bar{\boldsymbol{p}}) = \sum_{i=1}^{n+1} a_i \boldsymbol{p}_i + \boldsymbol{c} = \boldsymbol{0}, \text{and } \sum_{i=1}^{n+1} a_i = 0 \quad (24)$$

where $a_i \in \mathbb{R}$, $i = 1, \ldots, n+1$, $\boldsymbol{0} \neq \boldsymbol{c} \in \mathbb{R}^3$, is not invariant under translation and scaling. Due to the bias term $\boldsymbol{c}$, the estimated target point will have an offset depending on both the initial value of $\hat{\boldsymbol{p}}(0)$ and $\boldsymbol{c}$.

Finally, consider the class of bearing-only dependent constraints

$$\boldsymbol{f}(\bar{\boldsymbol{p}}) = \boldsymbol{f}(\bar{\boldsymbol{b}}) \quad (25)$$

where $\bar{\boldsymbol{b}} = [\ldots, \boldsymbol{b}_{ij}^{\mathsf{T}}, \ldots]^{\mathsf{T}}$ such that $(v_i, v_j) \in \bar{\mathcal{E}}$, we also have the following invariant theorem.

Theorem 3.8: Constraint (25) is invariant with respect to a translation and a scaling of the whole framework.

Proof: Due to the IBR property of $\bar{\mathcal{G}}(\bar{\boldsymbol{p}})$ and according to Theorem 3.1, we have $\boldsymbol{g}_{ij}^* = \boldsymbol{g}_{ij}, \forall (v_i, v_j) \in \bar{\mathcal{E}}$. It follows that $\boldsymbol{f}(\bar{\boldsymbol{p}}^*) = \boldsymbol{f}(\bar{\boldsymbol{b}}^*) = \boldsymbol{f}(\bar{\boldsymbol{b}}) = \boldsymbol{f}(\bar{\boldsymbol{p}})$. Thus, the invariant properties of (25) are trivially satisfied. ■

IV. Decentralized Bearing-Based Solutions to the FWLP

In this section, we propose a strategy to solve the well-known FWLP in a decentralized manner based on only bearing vector measurements. We first introduce and reformulate the FWLP into a decentralized pointing consensus setup. Then, two decentralized solutions to the FWLP will be proposed based on the strategy (5)–(7) in the previous section. Finally, we provide analysis on convergence of the proposed control laws.

A. Fermat–Weber Location Problem

Consider n noncollocated points at $\boldsymbol{p}_i \in \mathbb{R}^3, \forall i \in \mathcal{I}$. For a set of positive weights $\omega_i > 0, \forall i \in \mathcal{I}$, the FWLP [23] is stated as follows: "Find the point in $\mathbb{R}^3$ that minimizes the weighted distance sum $f(\boldsymbol{q}) = \sum_{i=1}^n \omega_i \|\boldsymbol{q} - \boldsymbol{p}_i\|$." Equivalently, it is required to find $\boldsymbol{q}^* \in \mathbb{R}^3$ such that

$$\boldsymbol{q}^* = \arg\min_{\boldsymbol{q} \in \mathbb{R}^3} f(\boldsymbol{q}). \quad (26)$$

The minimum $\boldsymbol{q}^*$ is often called the *Fermat–Weber point* of the set of n given points. If $\omega_i = 1, \forall i \in \mathcal{I}$, the solution $\boldsymbol{q}^*$ of (26) is called the geometric median of n points. A lot of studies related to the FWLP can be found in the literature, see [19]–[21], [49], for example. The following lemma is about the existence and uniqueness of the solution of the FWLP.

Lemma 4.1: [19, Th. 1] There exists a unique $\mathbf{q}^*$ minimizing the function $f(\boldsymbol{q})$. This minimum is characterized by the following optimality conditions.

1) If there exists $\boldsymbol{q}^*$, different from all $\boldsymbol{p}_i$, $i \in \mathcal{I}$, for which

$$\sum_{i=1}^n \omega_i \frac{\boldsymbol{q}^* - \boldsymbol{p}_i}{\|\boldsymbol{q}^* - \boldsymbol{p}_i\|} = \boldsymbol{0} \quad (27)$$

then this $\boldsymbol{q}^*$ is the minimum.

2) If for some $j \in \mathcal{I}$ we have

$$\left\| \sum_{i=1; i \neq j}^n \omega_i \frac{\boldsymbol{p}_j - \boldsymbol{p}_i}{\|\boldsymbol{p}_j - \boldsymbol{p}_i\|} \right\| \leq \omega_j \quad (28)$$

then this $\boldsymbol{p}_j$ is the minimum.

B. Decentralized Formulation of the FWLP

Consider a multiagent system consisting of n individual agents satisfying Assumptions 2.1 and 2.2. We further make the following assumption on the solution of the FWLP.

Assumption 4.1: Each agent i is given a strictly positive scalar weight ω_i. The unique minimum $\boldsymbol{q}^* \in \mathbb{R}^3$ of $f(\boldsymbol{q})$ satisfies condition (27).

Remark that Assumption 4.1 is to assure that the Fermat–Weber point $\boldsymbol{q}^*$ is located in the convex hull of $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$ and noncollocated with all $\boldsymbol{p}_i, i \in \mathcal{I}$. We can now state a decentralized pointing consensus formulation of the FWLP as follows.

Problem 4.1: Given an n-agent system satisfying Assumptions 2.1, 2.2, and 4.1, design a decentralized control law using only bearing information such that all agents' headings $\boldsymbol{b}_i, \forall i \in \mathcal{I}$, asymptotically point toward the solution $\boldsymbol{q}^*$ of the FWLP.

In other words, we would like to solve a pointing consensus problem when the Fermat–Weber point is the common target, i.e., $\boldsymbol{b}_i \to \boldsymbol{b}_i^* = \frac{\boldsymbol{q}^* - \boldsymbol{p}_i}{\|\boldsymbol{q}^* - \boldsymbol{p}_i\|}, \forall i \in \mathcal{I}$, where $\boldsymbol{b}_i^*$ satisfy $\sum_{i=1}^n \omega_i \boldsymbol{b}_i^* = \mathbf{0}$.

In Problem 4.1, constraint (27) depends only on the bearing vectors. Thus, constraint (27) belongs to the class of constraints (25). Let $\boldsymbol{p}_{n+1} = \boldsymbol{q}^*$ and $a_i = \frac{\omega_i}{\|\boldsymbol{p}_{n+1} - \boldsymbol{p}_i\|}, \forall i \in \mathcal{I}$, we can rewrite (27) as follows:

$$\sum_{i=1}^n \frac{\omega_i}{\|\boldsymbol{p}_{n+1} - \boldsymbol{p}_i\|}(\boldsymbol{p}_{n+1} - \boldsymbol{p}_i) = \mathbf{0}$$

$$\text{or,} \qquad \sum_{i=1}^n a_i \boldsymbol{p}_i - \left(\sum_{i=1}^n a_i\right) \boldsymbol{p}_{n+1} = \mathbf{0}. \tag{29}$$

By denoting $a_{n+1} = -\sum_{i=1}^n a_i$, it follows that $\sum_{i=1}^{n+1} a_i = 0$. Equation (29) has form of constraint (22), which shows that the Fermat–Weber point is in the convex hull of $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$ [21]. However, since $a_i = a_i(\boldsymbol{p}_i, \boldsymbol{p}_{n+1})$ depends on the positions, (29) does not belong to class (22).

C. Proposed Solutions

In the literature, a well-known solution to the FWLP is Weiszfeld's algorithm, which is a discrete-time iterative algorithm [19], [20]. Weiszfeld's algorithm is centralized since it requires information of all positions $\boldsymbol{p}_i, \forall i \in \mathcal{I}$. A continuous-time control law to reach to the minimum using only local bearing measurements was introduced in [24]. Other than [24], we are not aware of any other decentralized solutions to the FWLP in the literature.

This section proposes two decentralized solutions for pointing toward the minimum of the function f. First, all agents estimate their positions $\hat{\boldsymbol{p}}_i$ under the control law (5). After the position estimation step is at steady state ($\hat{\boldsymbol{p}}_i = \hat{\boldsymbol{p}}_i^*, \forall i \in \mathcal{I}$),[4] instead of (6), the agents adopt the estimated target point $\hat{\boldsymbol{q}}$ by a decentralized version of Weiszfeld's algorithm or the gradient-descent control law in [24]. Finally, the agents control their headings under the control law (7).

[4]Note that we can modify control law (5) as in [50] so that $\hat{\boldsymbol{p}}(t) = \hat{\boldsymbol{p}}^*$ after a finite time.

We now present two decentralized algorithms to the FWLP. The algorithms are hybrid in the sense that the agents eventually update the Fermat–Weber point in a series of event times $t_0, t_1, t_2, \ldots$, where $t_k = k\Delta T, k \geq 0$, and ΔT is a preselected positive number [28]. For brevity, we will adopt the notation $x[k] \equiv x(k\Delta T) \equiv x(t_k)$.

Algorithm 1: The decentralized Weiszfeld's algorithm.

1) Initially, all agents have the same estimation of the Fermat–Weber point: $\hat{\boldsymbol{q}}_i[0] = \hat{\boldsymbol{q}}[0] \neq \hat{\boldsymbol{p}}_i^*, \forall i \in \mathcal{I}$.
2) At step $k \geq 0$, we need to estimate two quantities

$$\bar{\boldsymbol{r}}[k] = \frac{1}{n}\sum_{i=1}^n \omega_i \frac{\hat{\boldsymbol{p}}_i^*}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|} \tag{30}$$

$$\bar{r}[k] = \frac{1}{n}\sum_{i=1}^n \frac{\omega_i}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|}. \tag{31}$$

Since correct estimations are required before moving to the step $k+1$, we employ the following finite-time consensus protocol for estimating the quantities $\bar{\boldsymbol{r}}[k]$ and $\bar{r}[k]$:

$$\dot{\boldsymbol{x}}_i(t) = k_x \sum_{j \in \mathcal{N}_i} \mathbf{sig}\left(\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t)\right)^\alpha. \tag{32}$$

Here, $0 < \alpha < 1$ is a parameter required for finite-time convergence, $k_x > 0$ is a control gain, and $t \in [t_k, t_{k+1})$. Let T be the convergence time of the algorithm, it is required that $\Delta T > T$. To estimate the vector $\bar{\boldsymbol{r}}[k]$, in (32), we initialize $\boldsymbol{x}_i(t_k) = \boldsymbol{r}_i(t_k) = \omega_i \frac{\hat{\boldsymbol{p}}_i^*}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|}, \forall i \in \mathcal{I}$. Meanwhile, to estimate the scalar $\bar{r}[k]$, we set $x_i(t_k) = r_i(t_k) = \frac{\omega_i}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|}, \forall i \in \mathcal{I}$. Then, for $t_k + T \leq t < t_{k+1}$, the consensus dynamics have settled to their averages, i.e., $\boldsymbol{r}_i(t) = \bar{\boldsymbol{r}}[k]$ and $r_i(t) = \bar{r}[k]$, and at $t = t_{k+1}$, each agent updates the estimation

$$\hat{\boldsymbol{q}}_i[k+1] = \frac{\boldsymbol{r}_i(t)}{r_i(t)} = \frac{\bar{\boldsymbol{r}}[k]}{\bar{r}[k]} = \frac{\sum_{i=1}^n \omega_i \frac{\hat{\boldsymbol{p}}_i^*}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|}}{\sum_{i=1}^n \frac{\omega_i}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|}} \tag{33}$$

which is precisely the formula of Weiszfeld's algorithm in [19] and [20].
3) Let $k \leftarrow k+1$ and repeat that procedure.

Algorithm 2: The decentralized gradient-descent algorithm.

1) Initially, all agents have the same estimation of the minimum point: $\hat{\boldsymbol{q}}_i[0] = \hat{\boldsymbol{q}}[0] \neq \hat{\boldsymbol{p}}_i^*, \forall i \in \mathcal{I}$.
2) At step $k \geq 0$, we first estimate the quantity

$$\bar{\boldsymbol{r}}[k] = \frac{1}{n}\sum_{i=1}^n \omega_i \frac{\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|}$$

by employing the finite-time consensus protocol (32). For estimation of $\bar{\boldsymbol{r}}[k]$, each agent $i \in \mathcal{I}$ initializes $\boldsymbol{x}_i(t_k) = \boldsymbol{r}_i[k] = \omega_i \frac{\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|}$ in (32). After a time T, for $t_k + T \leq t < t_{k+1}$, the consensus dynamics converged to the average, i.e., $\boldsymbol{r}_i(t) = \bar{\boldsymbol{r}}[k]$. Then, at $t = t_{k+1}$, each agent updates its estimate

$$\hat{\boldsymbol{q}}_i[k+1] = \hat{\boldsymbol{q}}_i[k] + k_q \bar{\boldsymbol{r}}[k] = \hat{\boldsymbol{q}}_i[k] + \frac{k_q}{n}\sum_{i=1}^n \omega_i \frac{\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|} \quad (34)$$

where $k_q > 0$ is a constant update gain. It can be checked that (34) is the discrete-time version of the control law in [24].

3) Let $k \leftarrow k+1$ and repeat that procedure.

Due to Algorithms 1 and 2, the right-hand side of (7) is discontinuous. We rewrite (7) as follows:

$$\dot{\boldsymbol{b}}_i(t) = k_b \boldsymbol{P}_{\boldsymbol{b}_i}(\hat{\boldsymbol{q}}_i[k] - \hat{\boldsymbol{p}}_i^*) \quad (35)$$

for $t_k \le t < t_{k+1}$, $k = 0,1,2,\ldots$, and $k_b > 0$ is a constant control gain. Thus, our proposed Fermat–Weber pointing consensus strategies include the position estimation law (5), Algorithm 1 or 2, and the heading vector control law (35).

D. Stability Analysis

In this section, we will prove that two control strategies 1) (5), Algorithm 1, (35), and 2) (5), Algorithm 2, (35) asymptotically solve Problem 4.1. In both strategies, the finite-time consensus protocol (32) is crucial for updating the estimation in Algorithms 1 and 2. The following lemmas are employed for the finite-time convergence analysis.

Lemma 4.2 (see [51]): If $\xi_1,\ldots,\xi_d \ge 0$ and $0 \le p \le 1$, then

$$\left(\sum_{i=1}^d \xi_i\right)^p \le \sum_{i=1}^d \xi_i^p.$$

Lemma 4.3 (see [29]): Suppose there exists a continuous function $V(\boldsymbol{x}) : \mathcal{D} \to \mathbb{R}$ such that the following conditions hold.

1) $V(\boldsymbol{x})$ is positive definite.
2) If there exist $\kappa > 0$, $\alpha \in (0,1)$, and an open neighborhood $\mathcal{U}_0 \in \mathcal{D}$ of the origin such that

$$\dot{V}(\boldsymbol{x}) + \kappa (V(\boldsymbol{x}))^\alpha \le 0 \quad \forall \boldsymbol{x} \in \mathcal{U}_0 \backslash \{\boldsymbol{0}\}$$

then $V(\boldsymbol{x})$ will reach zero in finite time with the settling time $T \le V(0)^{1-\alpha}/(\kappa(1-\alpha))$.

We have the following result on the finite-time consensus protocol (32).

Lemma 4.4: Under the control law (32), for each step $k \ge 0$, $\dot{\boldsymbol{x}}_i(t) \to \frac{1}{n}\sum_{i=1}^n \alpha_i \boldsymbol{b}_i[t_k], \forall i \in \mathcal{I}$, in a finite time T satisfying

$$T \le \frac{V(t_k)^{1-\alpha/2}}{\kappa(1-\alpha/2)} = \frac{2V(t_k)^{(2-\alpha)/2}}{\kappa(2-\alpha)}. \quad (36)$$

Proof: Although Lemma 4.4 can be considered as a corollary of the result in [22], it is not straightforward to derive an upper bound of the convergence time T of the control law (32) from [22]. Thus, we provide the proof of this lemma in the Appendix. ∎

Lemma 4.5: In each estimation step k of Algorithm 2

$$\boldsymbol{x}_i(t) = \bar{\boldsymbol{r}}[k] = \frac{1}{n}\sum_{i=1}^n \omega_i \frac{\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|} \quad \forall i \in \mathcal{I}$$

for $t_k + T \le t < t_{k+1}$, where T satisfies

$$T \le 2\left(2\sum_{i=1}^n \omega_i^2\right)^{\frac{2-\alpha}{2}} \Big/ (\kappa(2-\alpha)). \quad (37)$$

Proof: Let $\boldsymbol{\beta}_i[k] = \frac{\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|}, \forall i \in \mathcal{I}$, it follows that $\|\boldsymbol{\beta}_i[k]\| = 1$ and $\boldsymbol{x}_i(t_k) = \omega_i \boldsymbol{\beta}_i[k]$. Let $\bar{\boldsymbol{x}}(t_k) = \frac{1}{n}\sum_{i=1}^n \boldsymbol{x}_i(t_k)$, we have

$$\|\boldsymbol{\delta}_i(t_k)\|^2 = \|\boldsymbol{x}_i(t_k) - \bar{\boldsymbol{x}}(t_k)\|^2 \le 2(\|\boldsymbol{x}_i(t_k)\|^2 + \|\bar{\boldsymbol{x}}(t_k)\|^2) = 2\left(\omega_i^2 + \left\|\sum_{i=1}^n \omega_i \boldsymbol{\beta}_i[k]\right\|^2 \Big/ n^2\right).$$

Moreover

$$\left\|\sum_{i=1}^n \omega_i \boldsymbol{\beta}_i[k]\right\|^2 \le \sum_{i=1}^n \omega_i^2 \sum_{i=1}^n \|\boldsymbol{\beta}_i[k]\|^2 = n\sum_{i=1}^n \omega_i^2.$$

Thus, it follows that $\|\boldsymbol{\delta}_i(t_k)\|^2 \le 2(\omega_i^2 + \sum_{i=1}^n \omega_i^2/n), \forall i \in \mathcal{I}$. Summing up these inequalities, we get

$$V(t_k) = \|\boldsymbol{\delta}(t_k)\|^2/2 \le 2\sum_{i=1}^n \omega_i^2 \quad \forall k = 0,1,2,\ldots. \quad (38)$$

Therefore, inequality (37) follows immediately from (38) and (36). ∎

Remark 4.1: The upper bound of T in (37) provides a conservative lower bound for choosing the time step in implementing Algorithm 2. This upper bound is independent of $\hat{\boldsymbol{p}}_i^*$ and $\hat{\boldsymbol{q}}_i$. We can implement Algorithm 2 with $\Delta T \ge 2\left(2\sum_{i=1}^n \alpha_i^2\right)^{2-\alpha}/(\kappa(2-\alpha))$ to guarantee that $\boldsymbol{x}_i[\Delta T] = \bar{\boldsymbol{x}}[\Delta T], \forall i \in \mathcal{I}$.

Unfortunately, we cannot find an explicit lower bound that is independent on $\hat{\boldsymbol{p}}_i^*$ for choosing the time step in Algorithm 1. However, if $\hat{\boldsymbol{q}}_i[k] \ne \hat{\boldsymbol{p}}_i^*$ for all time, it can be proved that there exists a lower bound for choosing the time step ΔT. Thus, when implementing Algorithm 1, it is recommended to choose ΔT sufficiently large depending on the size of the estimation $\hat{\boldsymbol{p}}_i$.

The following lemma is about the asymptotic convergence of the estimation $\hat{\boldsymbol{q}}_i[k]$.

Lemma 4.6: Under Algorithm 1 or 2, all agents' estimations $\hat{\boldsymbol{q}}_i[k]$ asymptotically converge to the minimum $\hat{\boldsymbol{q}}^*$ of $f(\hat{\boldsymbol{q}}) = \sum_{i=1}^n \omega_i \|\hat{\boldsymbol{q}} - \hat{\boldsymbol{p}}_i^*\|$ if $\hat{\boldsymbol{q}}_i[k] \ne \hat{\boldsymbol{p}}_i^*$ for all time.

Proof: The update law (33) is the same as Weiszfeld's algorithm. Thus, as shown in [19] and [20], under Algorithm 1, the estimation $\hat{\boldsymbol{q}}_i[k] \to \hat{\boldsymbol{q}}^*$ as $k\Delta T \to \infty$ for almost all initial conditions.

The update law (34) is a gradient-descent law of f. Since f is a strictly convex function for $\hat{\boldsymbol{q}}_i[k] \ne \hat{\boldsymbol{p}}_i^*, \forall i \in \mathcal{I}$, $\nabla f = \sum_{i=1}^n \omega_i \frac{\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|}$ is locally Lipschitz continuous (i.e., $\exists \kappa > 0$ s.t. $\nabla^2 f = \sum_{i=1}^n \omega_i \frac{\boldsymbol{P}_{(\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k])}}{\|\hat{\boldsymbol{p}}_i^* - \hat{\boldsymbol{q}}_i[k]\|} \le \kappa \sum_{i=1}^n \omega_i \boldsymbol{I}_3$ in any neighborhood of $\hat{\boldsymbol{q}}^*$ that does not contain $\{\hat{\boldsymbol{p}}_i^* | i \in \mathcal{I}\}$), and the minimum is unique, it follows that the estimation $\hat{\boldsymbol{q}}_i[k]$ converges to $\hat{\boldsymbol{q}}^*$ asymptotically if $\hat{\boldsymbol{q}}_i[k] \ne \hat{\boldsymbol{p}}_i^*, i \in \mathcal{I}, \forall k \ge 0$ with a small fixed step size k_q/n.

Finally, both algorithms fail to find the minimum if and only if $\hat{\boldsymbol{q}}_i[k] = \hat{\boldsymbol{p}}_i^*$ for some time. The set of points that the algorithm fails is a set of measure zero in $\mathbb{R}^3$ [19], [21]. ■

We can now state the main result of this section.

Theorem 4.1: Suppose that Assumptions 2.1, 2.2, and 4.1 hold. If $\hat{\boldsymbol{q}}_i[k] \neq \hat{\boldsymbol{p}}_i^*, i \in \mathcal{I}, \forall k \geq 0$, the two proposed control strategies solve Problem 4.1, i.e., all heading vectors asymptotically point toward the minimum point $\boldsymbol{q}^*$ of $f(\boldsymbol{q})$.

Proof: First, the position estimation dynamics gives a solution of $\hat{\boldsymbol{p}}_i^*$ for all cases. Second, Algorithms 1 and 2 asymptotically give the solution $\hat{\boldsymbol{q}}^*$.

It is remained to prove that the pointing dynamics (7) guides all the heading vectors to $\boldsymbol{q}^*$. Due to the implementation of Algorithms 1 and 2, the right-hand side of (7) is discontinuous at $t = k\Delta T$, where $k = 0, 1, 2, \ldots$. However, for any time t, $\boldsymbol{b}_i(t)^\top \dot{\boldsymbol{b}}_i(t) = \boldsymbol{b}_i(t)^\top \boldsymbol{P}_{\boldsymbol{b}_i}(\hat{\boldsymbol{q}}_i[k] - \hat{\boldsymbol{p}}_i^*) = \boldsymbol{0}$ and thus $\|\boldsymbol{b}_i(t)\| = 1$. It shows that during the evolution of $\hat{\boldsymbol{q}}_i[k]$, the trajectory of (7) is bounded and will not diverge. Thus, if $\hat{\boldsymbol{q}}_i[k] \to \hat{\boldsymbol{q}}^*$ asymptotically, by a similar argument as in Theorem 3.4

$$\boldsymbol{b}_i(t) \to \frac{\hat{\boldsymbol{q}}^* - \hat{\boldsymbol{p}}_i^*}{\|\hat{\boldsymbol{q}}^* - \hat{\boldsymbol{p}}_i^*\|}$$

as $t \to \infty, \forall i \in \mathcal{I}$, or the heading vectors asymptotically point toward the minimum point $\boldsymbol{q}^*$ of $f(\boldsymbol{q})$. ■

Remark 4.2: Until now, we have assumed that $\omega_i > 0, \forall i \in \mathcal{I}$, and condition (27) is satisfied (see Assumption 4.1). By incorporating a conditional statement into the algorithms, we may relax this assumption. For example, consider Algorithm 1 and suppose that condition (28) holds, i.e., the Fermat–Weber point is $\boldsymbol{p}_j$ for a $j \in \mathcal{I}$. At the beginning of the step k, each agent i first checks whether or not $\hat{\boldsymbol{q}}_i[k] = \hat{\boldsymbol{p}}_i^*$. If the condition is true, then agent i initiates $\boldsymbol{r}_i(t_k) = \boldsymbol{x}_i(t_k) = \boldsymbol{r}_i(t_{k-1})$, and $r_i(t_k) = x_i(t_k) = r_i(t_{k-1})$ before employing a finite-time consensus step in which it acts as the leader. After a time T, we have $r_j(t) = r_i(t_k)$, and $\boldsymbol{r}_j(t) = \boldsymbol{r}_i(t_k), \forall j \in \mathcal{I}$, and thus $\hat{\boldsymbol{q}}_j[k+1] = \hat{\boldsymbol{q}}_i[k+1] = \hat{\boldsymbol{p}}_i^*$. By this way, we generate the following sequence in a distributed manner:

$$\hat{\boldsymbol{q}}_i[k+1] = \begin{cases} \frac{\sum_{i=1}^n \frac{\omega_i \hat{\boldsymbol{p}}_i^*}{\|\hat{\boldsymbol{q}}_i[k] - \hat{\boldsymbol{p}}_i^*\|}}{\sum_{i=1}^n \frac{\omega_i}{\|\hat{\boldsymbol{q}}_i[k] - \hat{\boldsymbol{p}}_i^*\|}}, & \hat{\boldsymbol{q}}_i[k] \notin \{\hat{\boldsymbol{p}}_1^*, \ldots, \hat{\boldsymbol{p}}_n^*\} \\ \hat{\boldsymbol{p}}_j^*, & \hat{\boldsymbol{q}}_j[k] = \hat{\boldsymbol{p}}_j^* \text{ for some } j \in \mathcal{I} \end{cases}$$

and this sequence converges to the estimated Fermat–Weber point $\hat{\boldsymbol{q}}^*$ [21]. Then, under the heading dynamics (35), all agents asymptotically consent their headings to $\boldsymbol{p}_i$ and agent i's heading trivially intersects $\boldsymbol{p}_i$. Thus, a pointing consensus is achieved even if (28) holds. The flowchart of Algorithm 1 with this modification is given in Fig. 6. Note that the modified algorithm also fails to solve Problem 4.1 if $\hat{\boldsymbol{q}}_j[k] = \hat{\boldsymbol{p}}_j^*$ for some $\hat{\boldsymbol{p}}_j^*$ does not satisfy conditions (27)–(28).

V. Simulation Results

In this section, we conduct numerical simulations to verify our analysis in the previous sections. A simulation is about the centroid pointing consensus problem in Section III. The other simulations are about the Fermat–Weber location problem in Section IV.

A. Simulation 1: Pointing to the Group's Centroid

We consider a six-agent system whose information graph $\mathcal{G}$ is given in Fig. 2. Six agents are positioned at $\boldsymbol{p}_1 = [3, 0, \frac{9}{2}]^\top, \boldsymbol{p}_2 = [\frac{3}{2}, \frac{3\sqrt{3}}{2}, \frac{9}{2}]^\top, \boldsymbol{p}_3 = [0, 0, \frac{9}{2}]^\top, \boldsymbol{p}_4 = [3, 0, 0]^\top, \boldsymbol{p}_5 = [\frac{3}{2}, \frac{3\sqrt{3}}{2}, 0]^\top$, and $\boldsymbol{p}_6 = [0, 0, 0]^\top$, respectively. The position estimation control law (5) is chosen according to Remark 3.2, with the parameters given as follows: $\sigma_{12} = 0.10, \sigma_{13} = 0.15, \sigma_{14} = 0.05, \sigma_{16} = 0.02, \sigma_{23} = 0.03, \sigma_{34} = 0.07, \sigma_{45} = 0.075, \sigma_{46} = 0.11, \sigma_{56} = 0.065, \sigma = 0.20$, and $\rho = 0.01$. The initial estimates $\hat{\boldsymbol{p}}_i(0)$, $\hat{\boldsymbol{q}}_i(0) = \hat{\boldsymbol{p}}_i(0)$, and the initial heading directions $\boldsymbol{b}_i(0), \forall i \in \mathcal{I}$, were randomly generated.

We simulate the six-agent system under the control strategy (5)–(7). Simulation results are given in Figs. 7(a) and (b), and 8. From Fig. 7(a), it can be observed that the estimated positions asymptotically take up a configuration $\hat{\boldsymbol{p}}$ different from the real configuration $\boldsymbol{p}$ by a translation and a dilation. Meanwhile, under the consensus protocol (6), $\hat{\boldsymbol{q}}_i, i \in \mathcal{I}$, asymptotically converges to the estimated centroid $\hat{\boldsymbol{p}}_c$ of $\{\boldsymbol{p}_1^*, \ldots, \boldsymbol{p}_6^*\}$. This leads $\hat{\boldsymbol{q}}_i - \hat{\boldsymbol{p}}_i$ to gradually align with the direction from $\boldsymbol{p}_i$ to $\boldsymbol{p}_c$. Consequently, after about 20 s, the heading vectors of six agents concurrently target the group centroid as depicted in Fig. 8(a)–(g).

Thus, simulation results are consistent with the analysis in Section III.

B. Simulation 2: Pointing to the Fermat–Weber Point

We use the same six-agent system as in Simulation 1. The simulations in this section are conducted after each agent has already had an estimation $\hat{\boldsymbol{p}}_i^*$. The parameters of the cost function f are chosen as follows: $\omega_1 = \omega_2 = \omega_3 = \frac{1}{5}, \omega_4 = \omega_5 = \omega_6 = \frac{1}{8}$. We simulate the system under two control strategies to compare their performance.

Simulation 2a (Algorithm 1): The initial estimate of the Fermat–Weber point is chosen to be $\hat{\boldsymbol{q}}_i[0] = [12.9669, 1.2199, 7.7389]^\top$, which corresponds to the initial states $\boldsymbol{r}_i[0] = [0.8466, 0.0796, 0.5052]^\top$ and $r_i[0] = 0.0653$, $\forall i \in \mathcal{I}$. The chosen control gains are $k_x = 0.15$ and $k_b = 0.50$. The time step between two updates of $\hat{\boldsymbol{q}}_i[k]$ in Algorithm 1 is $\Delta T = 5$ s.

Simulation results are given in Fig. 9. The agents can point the heading vectors very close to the Fermat–Weber point after 40 s. Due to the fast convergence of Algorithm 1, the agents can approximate the direction to the Fermat–Weber point in a short time. After 20 s (three updates of Algorithm 1), the agents point to $\boldsymbol{q}_i[3] = [1.6105, 0.8606, 2.9308]^\top$, which is quite close to the Fermat–Weber point ($[1.500, 0.8666, 3.3451]^\top$). It is also observed that the pointing dynamics (35) is able to track the target in the interval between two updates.

Simulation 2b (Algorithm 2): The initial estimate of the Fermat–Weber point is also chosen to be $\hat{\boldsymbol{q}}_i[0] = [12.9669, 1.2199, 7.7389]^\top$ as in Simulation 2a. The time step between two updates of $\hat{\boldsymbol{q}}_i[k]$ in Algorithm 2 is $\Delta T = 5$ s. The chosen control gains are $k_x = 0.15$ and $k_b = 0.50$, and the update rate in (34) is $k_q = 6$.

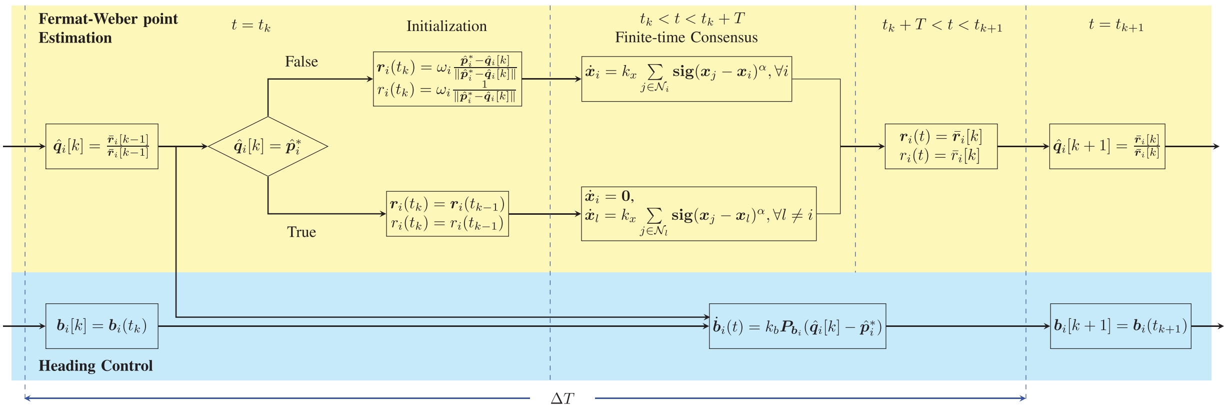

Fig. 6. Modified algorithm to solve Problem 4.1 without Assumption 4.1.

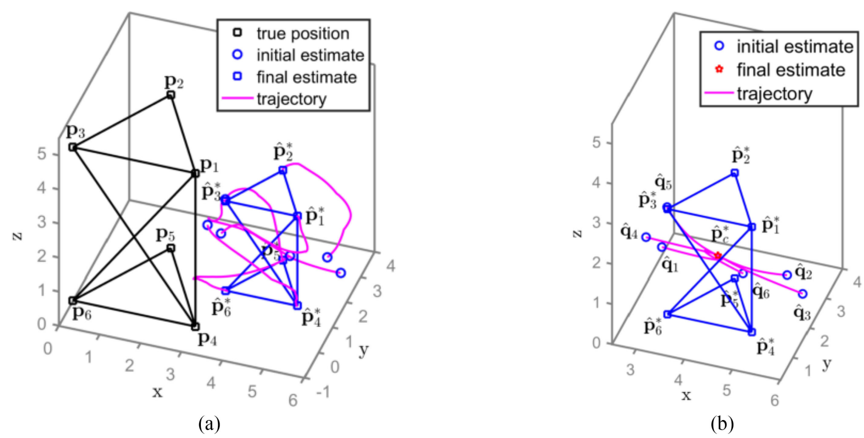

Fig. 7. Simulation 1. (a) Trajectories of $\hat{\boldsymbol{p}}_i(t), \forall i \in \mathcal{I}$, under the position estimation law (5). The final estimated configuration $\hat{\boldsymbol{p}}^*$ is different from the true configuration $\boldsymbol{p}$ by a translation and a scaling. (b) Trajectories of $\hat{\boldsymbol{q}}_i, \forall i \in \mathcal{I}$, under the consensus protocol (6). All $\hat{\boldsymbol{q}}_i(t), i \in \mathcal{I}$, asymptotically converge to $\hat{\boldsymbol{p}}_c^*$—the group's centroid of both $\{\hat{\boldsymbol{p}}_i(0)\}_{i\in\mathcal{I}}$ and $\{\hat{\boldsymbol{p}}_i^*\}_{i\in\mathcal{I}}$.

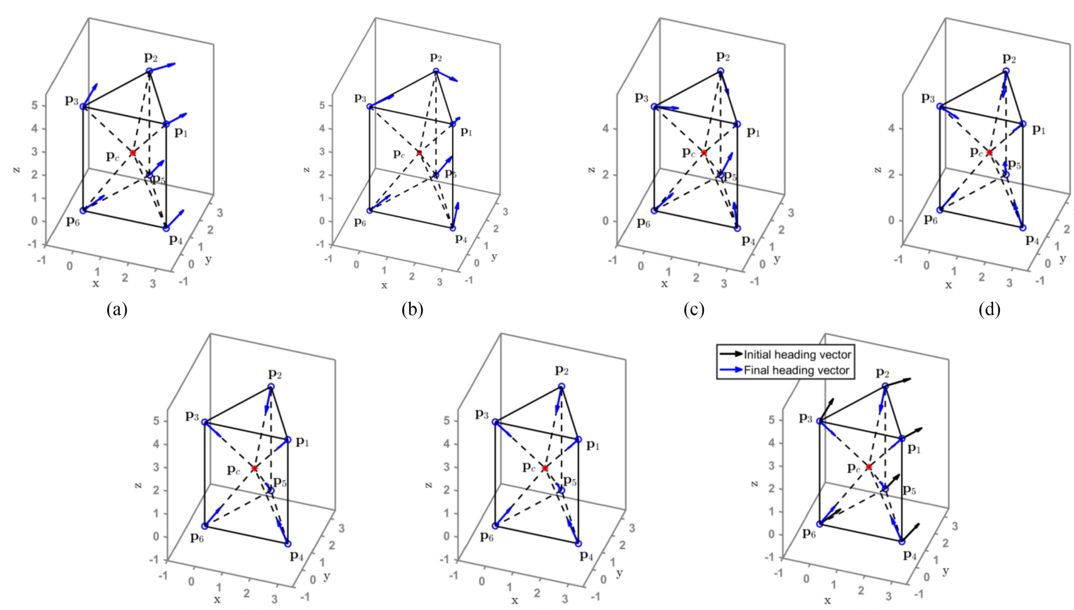

Fig. 8. Simulation 1: The heading vectors of six agents under the proposed control laws (5)–(7). All heading vectors asymptotically point toward the group's centroid. (a) t = 0 s. (b) t = 0.5 s. (c) t = 1 s. (d) t = 2 s. (e) t = 5 s. (f) t = 10 s. (g) t = 20 s.

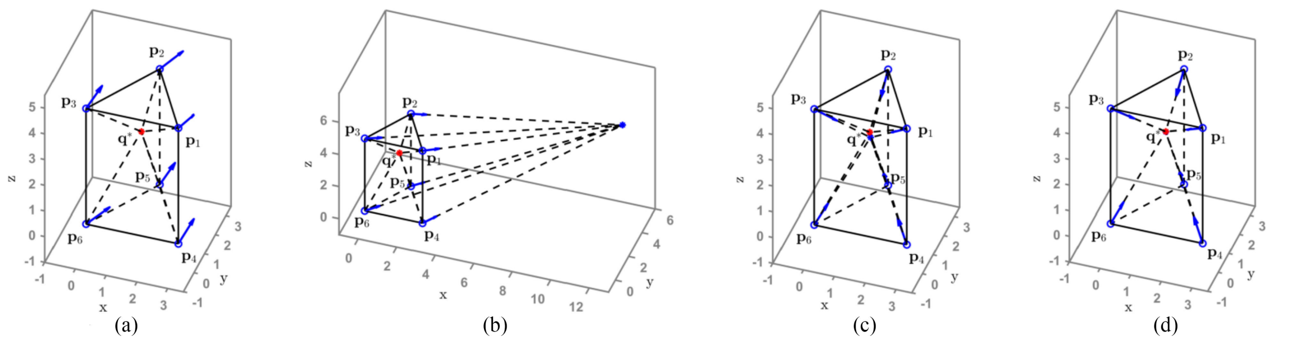

Fig. 9. Simulation 2a: The heading vectors of six agents under the proposed control laws (5), Algorithm 1, (35). (a) $t = 0$ s. (b) $t = 10$ s. (c) $t = 20$ s. (d) $t = 40$ s.

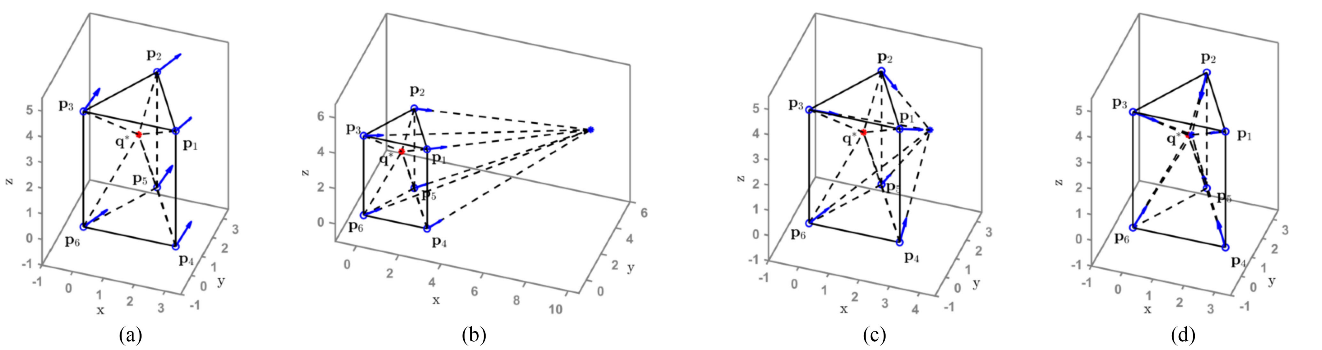

Fig. 10. Simulation 2b: The heading vectors of six agents under the proposed control laws (5), Algorithm 2, (35). (a) $t = 0$ s. (b) $t = 20$ s. (c) $t = 60$ s. (d) $t = 100$ s.

Simulation results are given in Fig. 10. After about 100 s, the agents' heading vectors point to $\boldsymbol{q}_i[19] = [1.6716, 0.8188, 3.4373]^\mathsf{T}$, which is also quite close to the Fermat–Weber point. As can be observed from Fig. 10, the convergence rate of this strategy is much slower than the previous strategy.

We also observe that the convergence time of the first control strategy mostly depends on time step between two updates [convergence time of the finite-time consensus protocol (32)]. Meanwhile, the convergence time of the second control strategy mainly depends on the convergence time of Algorithm 2 [i.e., the gradient-descent update law (34)].

VI. Conclusion

This paper studied the weighted centroid pointing consensus problem in the 3-D space. The proposed solution was built up from the solutions of three subproblems: bearing-only network localization, target decision, and heading coordination. The bearing rigidity theory plays an important role in linking these subproblems together. Two decentralized solutions to the Fermat–Weber location problem were also proposed under this bearing-based pointing consensus setup.

For further studies, we would like to consider the problem with target point locating outside the convex hull of agents' positions. Another extension is studying the problem when all agents' local reference frames are not initially aligned. This formulation may be related to the bearing rigidity theories in $SE(2)$ and $SE(3)$ [52]–[54]. Moreover, if the desired target follows a time-varying trajectory, the pointing consensus problem becomes a cooperative target tracking problem, and the analysis will be, respectively, more complicated. Finally, suppose that all agents' local reference frames are not aligned, it is interesting to consider the inverse problem of aligning the agents' local coordinates if their headings had initially pointed to a same point.

Appendix
Proof of Lemma 4.4

At each step $k \geq 0$, we rewrite (32) as

$$\dot{\boldsymbol{x}}(t) = -k_x \bar{\boldsymbol{H}}^\mathsf{T} \mathbf{sig}(\bar{\boldsymbol{H}}\boldsymbol{x}(t))^\alpha \tag{39}$$

where $t_k \leq t < t_{k+1}$, and $\boldsymbol{x}(t) = [\boldsymbol{x}_1^\mathsf{T}, \ldots, \boldsymbol{x}_n^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{3n}$. Denote $\boldsymbol{\delta}_i(t) = \boldsymbol{x}_i(t) - \bar{\boldsymbol{x}}(t_k)$, where $\bar{\boldsymbol{x}}(t_k) = \left(\sum_{i=1}^n \boldsymbol{x}_i(t_k)\right)/n = (\mathbf{1}_n \otimes \boldsymbol{I}_3)^\mathsf{T}\boldsymbol{x}(t_k)/n$ is a constant vector for $t_k \leq t < t_{k+1}$, and let $\boldsymbol{\delta}(t) = [\boldsymbol{\delta}_1^\mathsf{T}, \ldots, \boldsymbol{\delta}_n^\mathsf{T}]^\mathsf{T}$. It follows that $\boldsymbol{\delta} = \boldsymbol{x} - \mathbf{1}_n \otimes \bar{\boldsymbol{x}}(t_k)$. Since $\boldsymbol{H}\mathbf{1}_n = \mathbf{0}$, we have $\bar{\boldsymbol{H}}\boldsymbol{x} = \bar{\boldsymbol{H}}\boldsymbol{\delta}$. Thus, we can write

$$\dot{\boldsymbol{\delta}} = -k_x \bar{\boldsymbol{H}}^\mathsf{T} \mathbf{sig}(\bar{\boldsymbol{H}}\boldsymbol{\delta})^\alpha. \tag{40}$$

Consider the Lyapunov function $V(t) = \frac{1}{2}\|\boldsymbol{\delta}\|^2$, which is positive definite, radially unbounded, and continuously differentiable. For $t_k \leq t < t_{k+1}$

$$\begin{aligned}\dot{V} &= -k_x \boldsymbol{\delta}^\mathsf{T} \bar{\boldsymbol{H}}^\mathsf{T} \mathbf{sig}(\bar{\boldsymbol{H}}\boldsymbol{\delta})^\alpha \\ &= -k_x \sum_{k=1}^{3m} |[\bar{\boldsymbol{H}}\boldsymbol{\delta}]_k|^\alpha \\ &= -k_x \sum_{k=1}^{3m} |[\bar{\boldsymbol{H}}\boldsymbol{\delta}]_k^2|^{\frac{\alpha}{2}} \quad (41) \\ &\leq -k_x \left(\sum_{k=1}^{3m} |[\bar{\boldsymbol{H}}\boldsymbol{\delta}]_k^2| \right)^{\frac{\alpha}{2}} \quad (42)\end{aligned}$$

where in (41), we have used the inequality in Lemma 4.2 to get (42). Thus

$$\begin{aligned}\dot{V} &\leq -k_x \left(\|\bar{\boldsymbol{H}}\boldsymbol{\delta}\|^2 \right)^{\frac{\alpha}{2}} = -k_x \left(\boldsymbol{\delta}^\mathsf{T} \bar{\boldsymbol{L}} \boldsymbol{\delta} \right)^{\frac{\alpha}{2}} \\ &\leq -k_x \left(2\lambda_2(\boldsymbol{L}) \right)^{\frac{\alpha}{2}} \left(\frac{\|\boldsymbol{\delta}\|^2}{2} \right)^{\frac{\alpha}{2}} = -\kappa V^{\frac{\alpha}{2}} \quad (43)\end{aligned}$$

where $\boldsymbol{L}$ is the Laplacian matrix of $\mathcal{G}$, $\bar{\boldsymbol{L}} = \boldsymbol{L} \otimes \boldsymbol{I}_3$, $\lambda_2(\boldsymbol{L}) > 0$ is the second smallest eigenvalue of $\boldsymbol{L}$ [2], and $\kappa = k_x \left(2\lambda_2(\boldsymbol{L})\right)^{\frac{\alpha}{2}}$. It follows from (43) and Lemma 4.3 that $\boldsymbol{\delta} \to \mathbf{0}$ or $\boldsymbol{x}_i \to \bar{\boldsymbol{x}}(t_k)$ in finite time T. Also, the convergence time is upper bounded by

$$T \leq \frac{V(t_k)^{1-\alpha/2}}{\kappa(1-\alpha/2)} = \frac{2V(t_k)^{(2-\alpha)/2}}{\kappa(2-\alpha)}.$$

■

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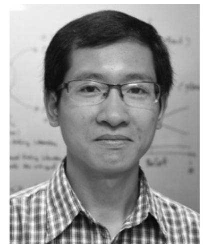

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