

On the Robustness of Uncertain Consensus Networks

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Abstract—This work considers the robustness of uncertain consensus networks. The stability properties of consensus networks with negative edge weights are also examined. We show that the network is unstable if either the negative weight edges form a cut in the graph or any single negative edge weight has a magnitude less than the inverse of the effective resistance between the two incident nodes. These results are then used to analyze the robustness of the consensus network with additive but bounded perturbations of the edge weights. It is shown that the small-gain condition is related again to cuts in the graph and effective resistance. For the single edge case, the small-gain condition is also shown to be exact. The results are then extended to consensus networks with nonlinear couplings.

Index Terms—Consensus protocol, graph theory, multiagent systems, network theory (graphs), robustness, uncertain systems.

I. INTRODUCTION

THE CONSENSUS protocol has recently emerged as a canonical model for the study of *networked dynamic systems*. In the linear setting, the consensus protocol has been studied from both dynamic systems and graph-theoretic perspectives [1]. The most basic setting considered in consensus networks assumes an undirected connected graph with non-negative weights on the edges of the graph. In such a setting, it is well known that the trajectories of all agents in the network converge to a common value. The use of edge weights in these networks often arises from the modeling of physical processes [2], or as a design parameter used to improve certain performance metrics [3], [4].

Recently, there has been a growing interest in multiagent networks containing negative edge weights. For example, in [4], it was shown that negative edge weights can appear as an optimal solution for finding the fastest converging linear iteration used in distributed averaging. Negative edge weights in problems related to the control of multiagent systems can also lead to steady-state configurations that are *clustering* [5], [6]. In [7], negative weights are used to model antagonistic interactions in a social network and conditions are provided for when such weights lead to *bipartite consensus*. The work of [8] provides bounds on the number of positive, negative,

and zero eigenvalues of the weighted Laplacian matrix with negative weights.

An important issue that has not received much attention is the *robustness* of these linear weighted consensus protocols. Indeed, even in the most basic setups, the trajectories generated by a weighted consensus protocol can be very rich, including steady-state trajectories that are synchronized, clustering, or unstable. If the weights in a consensus network arise from its physical modeling, then one must consider the performance of such systems when the weights are not known exactly. Similarly, if the weights are designed as an engineered parameter, then it is important to consider the robustness of the system in the presence of malicious attacks on the network, one of which could be the manipulation of the nominally designed edge weights.

The question of robustness in this context relates to the underlying graph. Recent works have analyzed the robustness of synchronization networks with random link failures [9], particular families of graphs in the context of the \mathcal{H}_2 performance [10], and uncertain tree structures [11]. In [12], uncertainties in the dynamics of each agent comprising a networked system were studied in the context of the small-gain theorem. However, these works do not address the fundamental *combinatorial* aspect that can arise in uncertain networked systems.

This then motivates the main contributions of this paper. We aim to study the robustness properties of linear weighted consensus protocols and provide graph-theoretic interpretations of the results. We consider consensus networks where the nominal edge weights are subject to bounded additive perturbations. At the heart of this analysis is an algebraic and graph-theoretic characterization of the definiteness of the weighted graph Laplacian with negative edge weights. We then apply these results to a linear consensus protocol in the context of the celebrated *small gain theorem*. We show that in certain cases, the small gain theorem provides an exact robustness margin for the uncertain consensus network. Furthermore, this measure turns out to be related to the notion of the *effective resistance* of a graph. These results are also extended to consider edge weights with nonlinear, but sector bounded perturbations.

The organization of this paper is as follows. In Section II, fundamental notions from graph theory are reviewed, including the results on the weighted graph and edge Laplacian matrices [13]–[15]. Section III introduces the uncertain consensus models. Section IV provides a general analysis on the stability of linear consensus protocols with negative edge weights, and Section V leverages these results to make statements on the robust stability of uncertain networks. Finally, a numerical example and concluding remarks are offered in Sections VI and VII.

Preliminaries: The notations employed in this paper are standard. For a real $n \times m$ matrix A , $\bar{\sigma}(A)$ denotes its largest

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TABLE I
SUMMARY OF THE MATRIX REPRESENTATIONS FOR GRAPHS

Incidence Matrix	$E(\mathcal{G}) = \begin{bmatrix} E_{\mathcal{F}} & E_{\mathcal{C}} \end{bmatrix}$	Graph Laplacian	$L(\mathcal{G}) = EW E^T$
Tucker Representation	$T_{(\mathcal{F}, \mathcal{C})} = L_e(\mathcal{F})^{-1} E_{\mathcal{F}}^T E_{\mathcal{C}}$	Edge Laplacian	$L_e(\mathcal{G}) = W^{\frac{1}{2}} E^T E W^{\frac{1}{2}}$
Cut-set Matrix	$R_{(\mathcal{F}, \mathcal{C})} = \begin{bmatrix} I & T_{(\mathcal{F}, \mathcal{C})} \end{bmatrix}$	Essential Edge Laplacian	$L_{\text{ess}}(\mathcal{F}) = L_e(\mathcal{F}) R_{(\mathcal{F}, \mathcal{C})} W R_{(\mathcal{F}, \mathcal{C})}^T$

singular value, and the spectral norm of the matrix A is $\|A\| = \bar{\sigma}(A)$. The image and null space of a matrix A are denoted as $\text{IM}[A]$ and $\mathcal{N}[A]$, respectively. The inertia of a real symmetric matrix A , denoted by the triple $s(A) = (n_+, n_-, n_0)$, is the number of positive, negative, and zero eigenvalues of the matrix. An important result on the inertia of a matrix is *Sylvester's Law of Inertia*, which states that all congruent symmetric matrices have the same inertia [16].¹ The m -dimensional space of piecewise continuous vector-valued square integrable functions is denoted as $\mathcal{L}_2^m[0, \infty)$. For a linear dynamical system described by the state-space model $\dot{x}(t) = Ax(t) + Bu(t)$ $y(t) = Cx(t) + Du(t)$, its transfer function is denoted as $G(s) = C(sI - A)^{-1}B + D$. The \mathcal{L}_2 -induced gain of the system described by $G(s)$ is determined by the \mathcal{H}_∞ system norm $\|G(s)\|_\infty = \sup_\omega \bar{\sigma}(G(j\omega))$ [17].

II. FORESTS, CYCLES, AND THE EDGE LAPLACIAN

In this section, we introduce the fundamental notions of spanning forests and cycles in graphs and present their associated matrix representations [18]. We consider undirected weighted graphs described by the triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ consisting of the node set \mathcal{V} , edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and weight function that maps each edge to a scalar value $\mathcal{W} : \mathcal{E} \rightarrow \mathbb{R}$. We often collect the weights of all the edges in a diagonal matrix $W \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ such that $[W]_{kk} = \mathcal{W}(k) = w_k$ where $e_k = (i, j) \in \mathcal{E}$.

The maximal acyclic subgraph of \mathcal{G} is referred to as a *spanning forest*, denoted by $\mathcal{F} = (\mathcal{V}, \mathcal{E}_{\mathcal{F}})$. If \mathcal{F} is connected, it is called a *spanning tree*, and is denoted as $\mathcal{T} = (\mathcal{V}, \mathcal{E}_{\mathcal{T}})$. Every graph \mathcal{G} can always be expressed as the union of a spanning tree (spanning forest) and another subgraph containing the remaining edges, that is, $\mathcal{G} = \mathcal{T} \cup \mathcal{C}$ ($\mathcal{G} = \mathcal{F} \cup \mathcal{C}$). The subgraph \mathcal{C} necessarily ‘‘completes cycles’’ in \mathcal{G} , and is defined as $\mathcal{C} = (\mathcal{V}, \mathcal{E}_{\mathcal{C}}) \subset \mathcal{G}$ with $\mathcal{E}_{\mathcal{C}} = \mathcal{E} \setminus \mathcal{E}_{\mathcal{T}}$ (similarly defined with a forest); we refer to this as the *cycle subgraph*.

The *incidence matrix* of a graph (with arbitrary edge orientation) $E(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ is such that for edge $e_k = (i, j) \in \mathcal{E}$, $[E(\mathcal{G})]_{i,k} = 1$, $[E(\mathcal{G})]_{j,k} = -1$, and $[E(\mathcal{G})]_{\ell,k} = 0$ for $\ell \neq i, j$. With an appropriate labeling of the edges, we can always express the incidence matrix as $E(\mathcal{G}) = [E(\mathcal{T}) \ E(\mathcal{C})]$ ($E(\mathcal{G}) = [E(\mathcal{F}) \ E(\mathcal{C})]$). An important property of the incidence matrix is that $E(\mathcal{G})^T \mathbf{1} = 0$ for any graph \mathcal{G} , where $\mathbf{1}$ is the vector of all ones, and that $\text{rk}[E(\mathcal{G})] = n - c$ for a graph with c -connected components. For a more compact notation, we will write $E := E(\mathcal{G})$, $E_{\mathcal{T}} = E(\mathcal{T})$, $E_{\mathcal{F}} = E(\mathcal{F})$, and $E_{\mathcal{C}} = E(\mathcal{C})$.

The weighted graph Laplacian of \mathcal{G} is defined as $L(\mathcal{G}) = EW E^T \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$. It is well known that for a weighted graph with only positive edge weights (i.e., $\mathcal{W} : \mathcal{E} \rightarrow \mathbb{R}_{\geq 0}$), the inertia of the Laplacian can be expressed as $s(L(\mathcal{G})) = (|\mathcal{V}| - c, 0, c)$,

where c is the number of connected components of \mathcal{G} [18]. Another symmetric matrix representation of a graph is the *edge Laplacian* [13]. For weighted graphs, we define the edge Laplacian as $L_e(\mathcal{G}) := W^{1/2} E^T E W^{1/2} \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$.

We now review some basic results relating the weighted edge Laplacian matrix to the graph Laplacian.

Proposition II.1: The weighted Laplacian matrix $L(\mathcal{G}) = EW E^T$ is similar to the matrix

$$\begin{bmatrix} L_e(\mathcal{F}) R_{(\mathcal{F}, \mathcal{C})} W R_{(\mathcal{F}, \mathcal{C})}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_c \end{bmatrix} \quad (1)$$

where \mathcal{G} has c connected components, $\mathcal{F} \subseteq \mathcal{G}$ is a spanning forest, and $R_{(\mathcal{F}, \mathcal{C})} = [I \ L_e(\mathcal{F})^{-1} E_{\mathcal{F}}^T E_{\mathcal{C}}]$.

Proof: For a spanning forest, one has $\text{rk}[E_{\mathcal{F}}] = n - c$ and, therefore, $L_e(\mathcal{F}) = E_{\mathcal{F}}^T E_{\mathcal{F}}$ is invertible. Define the transformation matrices

$$V = [E_{\mathcal{F}} (E_{\mathcal{F}}^T E_{\mathcal{F}})^{-1} \ N_{\mathcal{F}}], \quad V^{-1} = \begin{bmatrix} E_{\mathcal{F}}^T \\ N_{\mathcal{F}}^T \end{bmatrix}$$

where $\text{IM}[N_{\mathcal{F}}] = \text{span}[\mathcal{N}[E_{\mathcal{F}}^T]]$. It is straightforward to verify that the matrix in (1) is equal to $V^{-1} L(\mathcal{G}) V$. ■

The matrix $L_e(\mathcal{F}) R_{(\mathcal{F}, \mathcal{C})} W R_{(\mathcal{F}, \mathcal{C})}^T := L_{\text{ess}}(\mathcal{F})$ is referred to as the *essential edge Laplacian* [14]. (For a connected graph with spanning tree \mathcal{T} , we write $L_{\text{ess}}(\mathcal{T})$.) Note that the essential edge Laplacian is therefore a non-singular matrix, and the edge Laplacian of a spanning forest ($L_e(\mathcal{F})$) is always positive-definite. The matrix $L_e(\mathcal{F})^{-1} E_{\mathcal{F}}^T E_{\mathcal{C}} := T_{(\mathcal{F}, \mathcal{C})}$ is sometimes referred to as the *Tucker representation* of a graph [19]. The rows of the matrix $R_{(\mathcal{F}, \mathcal{C})}$ form a basis for the *cut space* of a graph [18] and we refer to it as the *cut-set matrix*. For a more in-depth discussion on the matrices $R_{(\mathcal{F}, \mathcal{C})}$ and $T_{(\mathcal{F}, \mathcal{C})}$, please see [13] and [14]. Note also that the matrix $L_e(\mathcal{F})^{-1} E_{\mathcal{F}}^T$ is the *left-inverse* of $E_{\mathcal{F}}$; we denote this matrix as $E_{\mathcal{F}}^L$. For the reader's convenience, we summarize the various matrix representations of graphs in Table I.

Proposition II.1 immediately leads to the following results on the inertia of the weighted Laplacian, the weighted edge Laplacian, and the essential edge Laplacian.²

Theorem II.2 ([15]): Assume \mathcal{G} has c -connected components and $s(L(\mathcal{G})) = (n_+, n_-, n_0)$. Then, $s(L_{\text{ess}}(\mathcal{F})) = (n_+, n_-, n_0 - c)$. Furthermore, $L_{\text{ess}}(\mathcal{F})$ has the same inertia as the matrix $R_{(\mathcal{F}, \mathcal{C})} W R_{(\mathcal{F}, \mathcal{C})}^T$.

Proof: Using the similarity transformation matrix $L_e(\mathcal{F})^{1/2}$, we have that $L_e(\mathcal{F}) R_{(\mathcal{F}, \mathcal{C})} W R_{(\mathcal{F}, \mathcal{C})}^T$ is similar to $L_e(\mathcal{F})^{1/2} R_{(\mathcal{F}, \mathcal{C})} W R_{(\mathcal{F}, \mathcal{C})}^T L_e(\mathcal{F})^{1/2}$. This matrix is congruent to $R_{(\mathcal{F}, \mathcal{C})} W R_{(\mathcal{F}, \mathcal{C})}^T$ and, thus, by Sylvester's Law of Inertia has the same inertia as $L_e(\mathcal{F}) R_{(\mathcal{F}, \mathcal{C})} W R_{(\mathcal{F}, \mathcal{C})}^T$. ■

¹A square matrix A is *congruent* to a square matrix B of the same dimension if there exists an invertible matrix S such that $B = S^T A S$.

²Observe that $L_{\text{ess}}(\mathcal{F})$ is similar to a symmetric matrix, justifying our slight abuse of terminology by referring to the inertia of $L_{\text{ess}}(\mathcal{F})$.

An immediate corollary of Theorem II.2 is that if $s(L(\mathcal{G})) = (n_+, n_-, n_0)$ and \mathcal{G} have c -connected components, then $s(R_{(\mathcal{F},c)} W R_{(\mathcal{F},c)}^T) = (n_+, n_-, n_0 - c)$.

The matrix $R_{(\mathcal{F},c)} W R_{(\mathcal{F},c)}^T$ turns out to be closely related to many combinatorial properties of a graph. Most important is its connection to the notion of the effective resistance of a graph, which will be discussed in Section IV. Theorem II.2 thus shows that studying the definiteness of the weighted Laplacian can be reduced to studying the matrix $R_{(\mathcal{F},c)} W R_{(\mathcal{F},c)}^T$ which contains in a more explicit way information on how the location and magnitude of negative weight edges influence spectral properties. This theme will be periodically revisited throughout this paper.

III. CONSENSUS AND THE EDGE AGREEMENT

In this paper, we consider the linear weighted consensus protocol [1] in the presence of exogenous finite-energy disturbances. The consensus dynamics over the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ can thus be expressed as

$$\Sigma(\mathcal{G}) : \begin{cases} \dot{x}(t) = -L(\mathcal{G})x(t) + v(t) \\ z(t) = E(\mathcal{G}_o)^T x(t). \end{cases} \quad (2)$$

Here, $v(t) = [v_1(t), \dots, v_n(t)]^T \in \mathcal{L}_2^2[0, \infty)$ is a finite-energy exogenous disturbance entering each agent in the system. The performance of the system is measured in terms of the energy of the vector $z(t) \in \mathbb{R}^{|\mathcal{E}_o|}$ capturing the relative states over a set of edges determined by the unweighted graph $\mathcal{G}_o = (\mathcal{V}, \mathcal{E}_o)$.³

With this model, we can consider how finite energy disturbances affect the asymptotic deviation of a subset of the states to an agreement value. This can be formally analyzed by considering the \mathcal{L}_2 -induced gain of the system, that is, its \mathcal{H}_∞ performance, $\|\Sigma(\mathcal{G})\|_\infty$ [13]. It is a well-established result that if the graph \mathcal{G} is connected and all weights are positive, then (2) reaches consensus, that is, $\lim_{t \rightarrow \infty} z(t) = 0$, for any output graph \mathcal{G}_o . Therefore, $z(t) \in \mathcal{L}_2^{|\mathcal{E}_o|}[0, \infty)$ and $\|\Sigma(\mathcal{G})\|_\infty$ is finite. However, this is not the case if $L(\mathcal{G})$ has multiple eigenvalues at the origin (and certainly if any eigenvalue of $L(\mathcal{G})$ is positive). Indeed, if \mathcal{G} is disconnected and $E(\mathcal{G}_o) = \bar{\mathcal{G}}$,⁴ then $z(t)$ is not a finite-energy signal. On the other hand, for the same example and choosing $\mathcal{G}_o \subseteq \mathcal{G}$, it can be verified that $z(t) \in \mathcal{L}_2^{|\mathcal{E}_o|}[0, \infty)$ (since each component of the graph will reach consensus).

Nevertheless, we would like to examine some notion of performance of (2) for any choice of output graph \mathcal{G}_o . In this direction, we can define a coordinate transformation for (2), $\tilde{x}(t) = S^{-1}x(t)$, with

$$S = \begin{bmatrix} (E_{\mathcal{F}}^L)^T & N_{\mathcal{F}} \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} E_{\mathcal{F}}^T \\ N_{\mathcal{F}}^T \end{bmatrix}$$

where $\mathbf{IM}[N_{\mathcal{F}}] = \mathbf{span}\{\mathcal{N}[E_{\mathcal{F}}^T]\}$. Thus, the state vector $\tilde{x}(t)$ can be partitioned into two components as $\tilde{x}(t) = [x_{\mathcal{F}}(t)^T \ x_a(t)^T]^T$, where $x_{\mathcal{F}}(t) = E_{\mathcal{F}}^T x(t)$ are the relative states over the edges forming the spanning forest of \mathcal{G} with

c -connected components, and $x_a(t) = N_{\mathcal{F}}^T x(t)$ correspond to modes in the direction of the all-ones vector across each component of \mathcal{G} . Applying this transformation to (2) leads to the following system:

$$\begin{cases} \begin{bmatrix} \dot{x}_{\mathcal{F}}(t) \\ \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} -L_{\text{ess}}(\mathcal{F}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_c \end{bmatrix} \begin{bmatrix} x_{\mathcal{F}}(t) \\ x_a(t) \end{bmatrix} + S^{-1}v(t) \\ z(t) = E(\mathcal{G}_o)^T \begin{bmatrix} (E_{\mathcal{F}}^L)^T & N_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} x_{\mathcal{F}}(t) \\ x_a(t) \end{bmatrix}. \end{cases} \quad (3)$$

In the new coordinate system, it is now straightforward to show that $\lim_{t \rightarrow \infty} z(t) = E(\mathcal{G}_o)^T N_{\mathcal{F}} x_a(0)$ and furthermore that $E(\mathcal{G}_o)^T (E_{\mathcal{F}}^L)^T x_{\mathcal{F}}(t) \in \mathcal{L}_2^{|\mathcal{E}_{\mathcal{F}}|}[0, \infty)$. We can now consider the following truncated system:

$$\Sigma_{\mathcal{F}}(\mathcal{G}) : \begin{cases} \dot{x}_{\mathcal{F}}(t) = -L_{\text{ess}}(\mathcal{F})x_{\mathcal{F}}(t) + E_{\mathcal{F}}^T v(t) \\ z_{\mathcal{F}}(t) = E(\mathcal{G}_o)^T (E_{\mathcal{F}}^L)^T x_{\mathcal{F}}(t). \end{cases} \quad (4)$$

We call the system $\Sigma_{\mathcal{F}}(\mathcal{G})$ the *edge agreement protocol* over the spanning forest $\mathcal{F} \subseteq \mathcal{G}$. The above transformation holds also when \mathcal{G} is connected and $\mathcal{F} = \mathcal{T}$ is a spanning tree. It is verifiable that the system $\Sigma_{\mathcal{F}}(\mathcal{G})$ is *minimal*. When $\mathcal{F} = \mathcal{T}$ is a spanning tree, then $\Sigma_{\mathcal{F}}(\mathcal{G})$ is a minimal realization of $\Sigma(\mathcal{G})$ [14].

A. Uncertain Edge Agreement Protocol

We now introduce a notion of uncertainty into the edge agreement protocols. First, we examine an uncertainty model where it is assumed that the exact weights of a subset of edges are an uncertain but bounded perturbation about some nominal value. In this direction, let $\mathcal{E}_\Delta \subseteq \mathcal{E}$ denote the set of uncertain edges. The nominal edge weight for an edge $k \in \mathcal{E}_\Delta$ is determined by the weight function \mathcal{W} . The uncertainty of the weight on edge k is modeled as an additive perturbation to the nominal edge weight as $w_k + \delta_k$ with $|\delta_k| \leq \bar{\delta}$ for some finite positive scalar $\bar{\delta}$. Thus, we can define the uncertainty set as

$$\mathbf{\Delta} = \{\Delta : \Delta = \mathbf{diag}\{\delta_1, \dots, \delta_{|\mathcal{E}_\Delta|}\}, \|\Delta\| \leq \bar{\delta}\}.$$

In this way, we can consider the uncertain edge agreement protocol as

$$\Sigma_{\mathcal{F}}(\mathcal{G}, \Delta) : \begin{cases} \dot{x}_{\mathcal{F}}(t) = -L_e(\mathcal{F})R_{(\mathcal{F},c)}(W + P\Delta P^T)R_{(\mathcal{F},c)}^T x_{\mathcal{F}}(t) \\ \quad + E_{\mathcal{F}}^T v(t) \\ z(t) = E(\mathcal{G}_o)^T (E_{\mathcal{F}}^L)^T x_{\mathcal{F}}(t). \end{cases} \quad (5)$$

for $\Delta \in \mathbf{\Delta}$. The matrix $P \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}_\Delta|}$ is a $\{0, 1\}$ -matrix used to select the uncertain edges with $[P]_{ij} = 1$ if $e_i \in \mathcal{E}_\Delta$, and $[P]_{ij} = 0$ otherwise [i.e., $E(\mathcal{G})P = E(\mathcal{G}_\Delta)$ with $\mathcal{G}_\Delta = (\mathcal{V}, \mathcal{E}_\Delta)$]. In this form, we see that the uncertainty is a structured additive uncertainty. This setup is visualized by the two-port block diagram in Fig. 1.

We also consider the consensus protocol with nonlinear couplings. By introducing appropriate assumptions on the nonlinear couplings, we are able to cast the problem as an uncertain

³Note that \mathcal{G}_o need not have any dependence on \mathcal{G} .

⁴The notation $\bar{\mathcal{G}}$ denotes the *complement* of the graph \mathcal{G} .

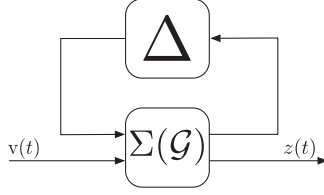


Fig. 1. Consensus network with uncertainties.

agreement protocol in the form of Fig. 1. In this direction, the nonlinear consensus protocol has the form

$$\begin{aligned}\dot{x}(t) &= -L(\mathcal{G})x(t) - E(\mathcal{G}_\Delta)\Phi(E(\mathcal{G}_\Delta)^T x(t)) + v(t) \\ z(t) &= E(\mathcal{G}_o)^T x(t).\end{aligned}$$

As before, we assume $\mathcal{E}_\Delta \subseteq \mathcal{E}$. The nonlinear vector function $\Phi: \mathbb{R}^{|\mathcal{E}_\Delta|} \rightarrow \mathbb{R}^{|\mathcal{E}_\Delta|}$ is assumed to be decoupled, that is, $\Phi(y) = [\phi_1(y_1), \dots, \phi_{|\mathcal{E}_\Delta|}(y_{|\mathcal{E}_\Delta|})]^T$. Furthermore, we assume that the nonlinear functions $\phi_i(\cdot)$ belong to the sector $[\alpha_i, \beta_i]$; that is, $\alpha_i u_i^2 \leq u_i \phi_i(y_i) \leq \beta_i u_i^2$ for all $u_i \in \mathbb{R}$ and $\alpha_i < \beta_i$ both real numbers. The corresponding nonlinear edge agreement protocol can thus be expressed as

$$\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi): \begin{cases} \dot{x}_{\mathcal{F}}(t) = -L_{\text{ess}}(\mathcal{F})x_{\mathcal{F}}(t) \\ \quad - L_e(\mathcal{F})R_{(\mathcal{F},\mathcal{C})}P\left(\Phi\left(P^T R_{(\mathcal{F},\mathcal{C})}^T x_{\mathcal{F}}(t)\right)\right) \\ \quad + E_{\mathcal{F}}^T v(t) \\ z(t) = E(\mathcal{G}_o)^T (E_{\mathcal{F}}^L)^T x_{\mathcal{F}}(t). \end{cases} \quad (6)$$

For both models $\Sigma_{\mathcal{F}}(\mathcal{G}, \Delta)$ and $\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi)$, we will be concerned with determining bounds on the uncertainty that guarantee the *robust stability* of the uncertain agreement protocols. The main analytic tool will be the application of the small gain theorem. To proceed with this analysis, we first examine stability properties of the linear consensus protocol with arbitrary negative edge weights. These results will then be applied to derive the more general robust stability statements of the uncertain models.

IV. ON THE STABILITY OF WEIGHTED CONSENSUS

Before examining the robust stability of the uncertain consensus networks presented in Section III-A, we first study the general stability properties of the linear consensus protocol $\Sigma(\mathcal{G})$ with negative weights. This section aims to reveal how the magnitude and placement of negative weight edges in a weighted graph can influence the stability properties of $\Sigma(\mathcal{G})$. In this section, we assume there are no exogenous disturbances entering the protocol (i.e., $v(t) = 0$).

Theorem II.2 revealed that the inertia of $L(\mathcal{G})$ is related to the inertia of $R_{(\mathcal{F},\mathcal{C})}^L W R_{(\mathcal{F},\mathcal{C})}^T$. We now exploit the structure of this matrix to show how negative edge weights affect the stability of the weighted consensus. We first examine the effect of negative edge weights for a class of graphs known as *signed graphs*. Then, we proceed to provide a general stability characterization that turns out to be related to the notion of the *effective*

resistance of a graph. Moreover, as $\Sigma(\mathcal{G})$ is a linear system, its stability can be determined by examining the eigenvalues of the weighted Laplacian $L(\mathcal{G})$. In particular, the stability of $\Sigma(\mathcal{G})$ is guaranteed as long as the weighted Laplacian is a positive semidefinite matrix. Therefore, the results to follow are presented in terms of the definiteness of $L(\mathcal{G})$.

A. Signed Graphs and the Stability of $\Sigma(\mathcal{G})$

A useful approach for the analysis of graphs with both positive and negative edge weights is the notion of *signed graphs* [20]. A signed graph is a weighted graph \mathcal{G} with a partition of the edge set into edges with positive weights, denoted as \mathcal{E}_+ , and edges with negative weights, denoted as \mathcal{E}_- . We can now define two subgraphs—one containing only positive weight edges, and one containing only negative weight edges; $\mathcal{G}_+ = (\mathcal{V}, \mathcal{E}_+, \mathcal{W}_+)$ and $\mathcal{G}_- = (\mathcal{V}, \mathcal{E}_-, \mathcal{W}_-)$. The weight maps \mathcal{W}_+ and \mathcal{W}_- are simply the original weight map \mathcal{W} restricted to either \mathcal{E}_+ or \mathcal{E}_- with corresponding diagonal weight matrices W_+ and W_- . The corresponding incidence matrices can also be written as [13]

$$\begin{aligned}E_+ &:= E(\mathcal{G}_+) = E_{\mathcal{F}_+} R_{(\mathcal{F}_+, \mathcal{C}_+)} = E_{\mathcal{F}_+} \begin{bmatrix} I & T_{(\mathcal{F}_+, \mathcal{C}_+)} \end{bmatrix} \\ E_- &:= E(\mathcal{G}_-) = E_{\mathcal{F}_-} R_{(\mathcal{F}_-, \mathcal{C}_-)} = E_{\mathcal{F}_-} \begin{bmatrix} I & T_{(\mathcal{F}_-, \mathcal{C}_-)} \end{bmatrix}.\end{aligned}$$

A signed graph is *structurally balanced* if and only if the vertex set can be partitioned into 2 sets \mathcal{V}_1 and \mathcal{V}_2 , such that all edges in \mathcal{E}_- connect nodes in \mathcal{V}_1 to nodes in \mathcal{V}_2 (i.e., \mathcal{G}_- is bipartite), and all edges in \mathcal{E}_+ connect nodes in \mathcal{V}_i only to nodes in \mathcal{V}_i for $i = 1, 2$ [21].

Observe that the weighted graph Laplacian can now be expressed in terms of the positive and negative weight subgraphs

$$L(\mathcal{G}) = E_+ W_+ E_+^T - E_- |W_-| E_-^T. \quad (7)$$

This decomposition leads to the following statement on the definiteness of the weighted Laplacian.

Theorem IV.1: The weighted Laplacian is positive semidefinite if and only if

$$\begin{bmatrix} |W_-|^{-1} & E_-^T \\ E_- & E_+ W_+ E_+^T \end{bmatrix} \geq 0. \quad (8)$$

Proof: This result follows from the Schur complement applied to (7). ■

The linear matrix inequality (LMI) in (8) already indicates that the magnitude of the negative edge weights plays an important role in the definiteness of the Laplacian. We now proceed to examine this LMI in more detail to reveal how also their location in the graph is important.

Corollary IV.2: The weighted Laplacian is positive semidefinite if and only if

$$\begin{bmatrix} |W_-|^{-1} & E_-^T (E_{\mathcal{F}_+}^L)^T & E_-^T N_{\mathcal{F}_+} \\ E_{\mathcal{F}_+}^L E_- & R_{(\mathcal{F}_+, \mathcal{C}_+)} W_+ R_{(\mathcal{F}_+, \mathcal{C}_+)}^T & 0 \\ N_{\mathcal{F}_+}^T E_- & 0 & \mathbf{0} \end{bmatrix} \geq 0. \quad (9)$$

Proof: Let $\mathbf{IM}[N_{\mathcal{F}_+}] = \mathbf{span}\{\mathcal{N}[E_{\mathcal{F}_+}^T]\}$. Consider the congruent transformation matrix

$$Q = \begin{bmatrix} I & \\ 0 & \left[\left(E_{\mathcal{F}_+}^L \right)^T N_{\mathcal{F}_+} \right] \end{bmatrix}.$$

Then the matrix in (8) has the same inertia (by the congruent transformation using Q) as the matrix in (9). From Theorem IV.1 it now follows that the weighted Laplacian is positive semi-definite if and only if (9) is positive semi-definite. ■

We can make some more observations about the above result. It directly follows that if \mathcal{G}_+ is connected, then $\mathbf{IM}[N_{\mathcal{F}_+}] = \mathbf{span}\{\mathbf{1}\}$, and $E_{\mathcal{F}_+}^T N_{\mathcal{F}_+} = 0$. Thus, $L(\mathcal{G}) \geq 0$ if and only if

$$\begin{bmatrix} |W_-|^{-1} & E_-^T \left(E_{\mathcal{F}_+}^L \right)^T \\ E_{\mathcal{F}_+}^L E_- & R_{(\mathcal{F}_+, \mathcal{C}_+)} W_+ R_{(\mathcal{F}_+, \mathcal{C}_+)}^T \end{bmatrix} \geq 0. \quad (10)$$

We are now prepared to make a statement connecting the location of the negative edge weights to the definiteness of the weighted Laplacian. First, we comment on the structural meaning of the matrix $E_{\mathcal{F}_+}^T N_{\mathcal{F}_+}$ in (9). The quantity $E_{\mathcal{F}_+}^T N_{\mathcal{F}_+}$ characterizes any cuts of the original graph \mathcal{G} using only the negative weighted edges. To show this, assume \mathcal{G}_+ has c -connected components and let $N_{\mathcal{F}_+} = [\mathbf{n}_1, \dots, \mathbf{n}_c]$. Then, \mathbf{n}_i is a $\{0, 1\}$ -vector with $[\mathbf{n}_i]_j = 1$ if and only if node $j \in \mathcal{V}$ is in component i . Then, it is clear that $E_-^T \mathbf{n}_i$ will be a $\{0, \pm 1\}$ -vector with $[E_-^T \mathbf{n}_i]_k = \pm 1$ if and only if $k = (u, v) \in \mathcal{E}_-$ such that u and v are not in the same components of \mathcal{G}_+ .

Theorem IV.3: Assume that \mathcal{G} is connected and $\mathcal{E}_+, \mathcal{E}_- \neq \emptyset$. If \mathcal{G}_+ is not connected, then the weighted Laplacian matrix is indefinite for any (nonzero) choice of negative edge weights.

Proof: Consider the following submatrix of (9) obtained by deleting the center block row and column

$$\begin{bmatrix} |W_-|^{-1} & E_-^T N_{\mathcal{F}_+} \\ N_{\mathcal{F}_+}^T E_- & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(m+c) \times (m+c)}$$

where $m = |\mathcal{E}_-|$ and c is the number of connected components of \mathcal{G}_+ . We assume that \mathcal{G}_+ is not connected and, thus, $N_{\mathcal{F}_+}$ contains c columns. Partition the matrix as $N_{\mathcal{F}_+} = [\mathbf{n}_1, \dots, \mathbf{n}_c]$ and recall that $[E_-^T \mathbf{n}_i]_k = \pm 1$ if and only if $k = (u, v) \in \mathcal{E}_-$ such that u and v are not in the same components of \mathcal{G}_+ . Denote by $\text{CUT}_i \subseteq \mathcal{E}_-$ as the set of negative weight edges used to form a cut separating the i th component of \mathcal{G}_+ from \mathcal{G} , and let $\text{CUT} = \cup_i \text{CUT}_i$. Then an expression for the quadratic form of the matrix of interest is

$$\begin{aligned} x^T \begin{bmatrix} |W_-|^{-1} & E_-^T N_{\mathcal{F}_+} \\ N_{\mathcal{F}_+}^T E_- & \mathbf{0} \end{bmatrix} x &= \sum_{i \in \mathcal{E}_-} |W_-(i)|^{-1} x_i^2 \\ &+ \sum_{k \in \text{CUT}_1} \pm 2x_k x_{m+1} + \dots \\ &+ \sum_{k \in \text{CUT}_c} \pm 2x_k x_{m+c}. \end{aligned}$$

From the quadratic form, it is now clear that the elements of the vector x_i for $i = m+1, \dots, m+c$ can be arbitrarily chosen to make the inequality negative. Therefore, there exists at

least one negative eigenvalue and the matrix in (9) is indefinite. From Corollary IV.2 we can conclude that the weighted graph Laplacian is indefinite independent of the value of the negative weights. ■

Theorem IV.3 shows that if any of the negative weight edges forms a cut in the graph, then the Laplacian matrix must have negative eigenvalues. A particular class of graphs satisfying the conditions of Theorem IV.3 are the structurally balanced signed graphs, immediately leading to the following corollary.

Corollary IV.4: If a signed graph \mathcal{G} is balanced, then $L(\mathcal{G})$ is indefinite for any choice of negative edge weights.

B. Effective Resistance and the Stability of $\Sigma(\mathcal{G})$

The main result of Section IV-A provides an analytical justification of what may be considered an intuitive result. That is, if the negative weight edges form a cut in the graph, then the weighted Laplacian will be indefinite, that is, $\Sigma(\mathcal{G})$ will be unstable. In this section, we reveal a more general condition on the negative edge weights that can lead to an indefinite weighted Laplacian. This condition turns out to be related to the notion of the *effective resistance* of a graph. Results from this section were recently reported in [15] and, thus, the reader is referred to that work for related proofs.⁵

It is well known that the weighted Laplacian of a graph can be interpreted as a resistor network [22]. Each edge in the network can be thought of as a resistor with a resistance equal to the inverse of the edge weight $r_k = \mathcal{W}(k)^{-1} = w_k^{-1}$ for $k \in \mathcal{E}$.⁶ The resistance between any two pairs of nodes can be determined using standard methods from electrical network theory [22]. It may also be computed using the Moore-Penrose pseudoinverse of the graph Laplacian, denoted as $L(\mathcal{G})^\dagger$.

Definition IV.5 ([22]): The effective resistance between nodes $u, v \in \mathcal{V}$ in a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is

$$\begin{aligned} \mathcal{R}_{uv}(\mathcal{G}) &= (\mathbf{e}_u - \mathbf{e}_v)^T L^\dagger(\mathcal{G}) (\mathbf{e}_u - \mathbf{e}_v) \\ &= [L^\dagger(\mathcal{G})]_{uu} - 2[L^\dagger(\mathcal{G})]_{uv} + [L^\dagger(\mathcal{G})]_{vv} \end{aligned}$$

where \mathbf{e}_u is the indicator vector for node u , that is, $\mathbf{e}_u = 1$ in the u position and 0 elsewhere.

Our first result shows how the effective resistance between two nodes is related to the matrix $R_{(\mathcal{T}, \mathcal{C})} W R_{(\mathcal{T}, \mathcal{C})}^T$ and the essential edge Laplacian.

Proposition IV.6 ([15]): Let \mathcal{G} be a connected graph and assume $s(L(\mathcal{G})) = (n_+, n_-, 1)$. Then

$$\begin{aligned} L^\dagger(\mathcal{G}) &= (E_{\mathcal{T}}^L)^T \left(R_{(\mathcal{T}, \mathcal{C})}^T W R_{(\mathcal{T}, \mathcal{C})} \right)^{-1} E_{\mathcal{T}}^L \\ &= (E_{\mathcal{T}}^L)^T L_{\text{ess}}(\mathcal{T})^{-1} E_{\mathcal{T}}^T. \end{aligned} \quad (11)$$

From Proposition IV.6, it is clear that the effective resistance between nodes $u, v \in \mathcal{V}$ can be expressed as

$$\mathcal{R}_{uv}(\mathcal{G}) = (\mathbf{e}_u - \mathbf{e}_v)^T (E_{\mathcal{T}}^L)^T \left(R_{(\mathcal{T}, \mathcal{C})}^T W R_{(\mathcal{T}, \mathcal{C})} \right)^{-1} E_{\mathcal{T}}^L (\mathbf{e}_u - \mathbf{e}_v). \quad (12)$$

⁵An extended version of this work containing all proofs is also available on arXiv (arXiv:1408.4471).

⁶Thus, the edge weight w_k can be interpreted as an *admittance*.

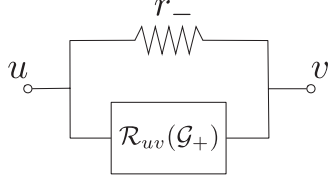


Fig. 2. Resistive network interpretation with one negative weight edge.

We now present a result showing that this equivalent characterization of the effective resistance is useful for understanding the definiteness of the weighted Laplacian.

Theorem IV.7 ([15]): Assume that \mathcal{G}_+ is connected and $|\mathcal{E}_-| = 1$ with $\mathcal{E}_- = \{e_- = (u, v)\}$. Let $\mathcal{R}_{uv}(\mathcal{G}_+)$ denote the effective resistance between nodes $u, v \in \mathcal{V}$ over the graph \mathcal{G}_+ . Then, $L(\mathcal{G})$ is positive semidefinite if and only if $|\mathcal{W}(e_-)| \leq \mathcal{R}_{uv}^{-1}(\mathcal{G}_+)$.

The above result has a very intuitive physical interpretation. The entire network \mathcal{G}_+ can be considered as a single lumped resistor between nodes u and v with resistance $\mathcal{R}_{uv}(\mathcal{G}_+)$. The negative-weight edge can thus be thought of as adding another resistor in parallel between the nodes, as in Fig. 2. The equivalent resistance between u and v is well known to be

$$\mathcal{R}_{uv}(\mathcal{G}) = \frac{\mathcal{R}_{uv}(\mathcal{G}_+)r_-}{\mathcal{R}_{uv}(\mathcal{G}_+) + r_-}.$$

If r_- is a negative resistor, then choosing $r_- = -\mathcal{R}_{uv}(\mathcal{G}_+)$ corresponds to an equivalent resistance that is infinite, that is, an *open circuit*. The open circuit can be thought of as a cut between the terminals u and v . This interpretation is inline also with the results of Theorem IV.3. Indeed, in this case, the graph \mathcal{G}_+ is not connected and the effective resistance between any two nodes connected by a negative edge weight over \mathcal{G}_+ is infinite, that is, $\mathcal{R}_{uv}^{-1}(\mathcal{G}_+) = 0$.

The result in Theorem IV.7 can be generalized to multiple negative weight edges with some additional assumptions on how those edges are distributed in the graph. In this direction, for each edge $k = (u, v) \in \mathcal{E}_-$, define the set $\mathcal{P}_k \subseteq \mathcal{E}_+$ to be the set of all edges in \mathcal{G}_+ that belong to a path connecting nodes u to v . Let $\mathcal{G}_+(\mathcal{P}_k) \subseteq \mathcal{G}_+$ be the subgraph induced by the edges in \mathcal{P}_k .⁷ Note that if $\mathcal{P}_k \cap \mathcal{P}_{k'} = \emptyset$ for edges with distinct nodes (i.e., $k = (u, v)$ and $k' = (u', v') \in \mathcal{E}_-$), then there exists no cycle in \mathcal{G}_+ containing the nodes u, v, u', v' . An important class of graphs that can admit such a partition is the *cactus graphs* [23]. Using this characterization, the following statement on effective resistance with multiple negative weight edges follows.

Theorem IV.8 ([15]): Assume that \mathcal{G}_+ is connected and $|\mathcal{E}_-| > 1$. Let $\mathcal{R}_k(\mathcal{G}_+)$ denote the effective resistance between nodes $u, v \in \mathcal{V}$ over the graph \mathcal{G}_+ with $k = (u, v) \in \mathcal{E}_-$, and let $\mathbf{R} = \text{diag}\{\mathcal{R}_1(\mathcal{G}_+), \dots, \mathcal{R}_{|\mathcal{E}_-|}(\mathcal{G}_+)\}$. Furthermore, assume that $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$ for all $i, j \in \mathcal{E}_-$, where \mathcal{P}_i is the set of all edges belonging to a path connecting the nodes incident to edge $i \in \mathcal{E}_-$. Then, the weighted Laplacian is positive semi-definite if and only if $|\mathcal{W}_-| \leq \mathbf{R}^{-1}$.

⁷Thus, $\mathcal{G}_+(\mathcal{P}_k) = (\mathcal{V}(\mathcal{P}_k), \mathcal{P}_k)$ where $\mathcal{V}(\mathcal{P}_k) \subseteq \mathcal{V}$ are the nodes incident to edges in \mathcal{P}_k .

Theorem IV.8 also has the same physical interpretation as Theorem IV.7. Indeed, the resistance between two nodes contained in a subgraph $\mathcal{G}_+(\mathcal{P}_k)$ is not determined by any other edges in the network. Both Theorems IV.7 and IV.8 provide a clear characterization of how negative weight edges can impact the definiteness of the weighted Laplacian, and how that is related to the effective resistance in the graph. In fact, from (10), we can also observe an additional fact relating the total effective resistance between all nodes incident to edges in \mathcal{E}_- and the definiteness of the graph, independent of the actual location of these edges in the network.

Corollary IV.9 ([15]): Assume that \mathcal{G}_+ is connected. If $L(\mathcal{G}) \geq 0$, then $\sum_{k \in \mathcal{E}_-} |\mathcal{W}(k)|^{-1} \geq \mathbf{R}_{\text{tot}}^{\mathcal{E}_-}$, where

$$\mathbf{R}_{\text{tot}}^{\mathcal{E}_-} = \text{trace} \left[E_-^T (E_{\mathcal{T}_+}^L)^T \left(R_{(\mathcal{T}_+, \mathcal{C}_+)} W_+ R_{(\mathcal{T}_+, \mathcal{C}_+)}^T \right)^{-1} E_{\mathcal{T}_+}^L E_- \right].$$

Corollary IV.9 indicates that a weighted Laplacian with negative weights can still be positive semidefinite, and in that case, the total magnitude of the negative weight edges is closely related to the total effective resistance in the network (defined over the nodes incident to \mathcal{E}_-). The notion of total effective resistance has also appeared in works characterizing the \mathcal{H}_2 performance of certain multiagent networks [24]–[26]. While Corollary IV.9 only provides a sufficient condition for the definiteness of the weighted Laplacian, it nevertheless reinforces its connection to the notion of effective resistance.

V. ON THE ROBUST STABILITY OF THE UNCERTAIN CONSENSUS PROTOCOL

The results of the previous section provide the correct foundation to now consider the robust stability of the uncertain consensus models presented in Section III-A. A natural approach for analyzing the robust stability is via the celebrated *small gain theorem*. While the small gain theorem often provides very conservative results, we demonstrate in this section that for certain classes of uncertainties, the small gain result is, in fact, an exact condition.

A. Robust Stability of $\Sigma_{\mathcal{F}}(\mathcal{G}, \Delta)$

We now proceed with an analysis of the system $\Sigma_{\mathcal{F}}(\mathcal{G}, \Delta)$ in (5). Based on the system interconnection shown in Fig. 1, the map from the exogenous inputs $v(t)$ to the controlled output $z(t)$ in the presence of the structured uncertainty $\Delta \in \mathbf{\Delta}$ can be characterized by the upper fractional transformation [17]

$$\bar{S}(\Sigma_{\mathcal{F}}(\mathcal{G}), \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} \quad (13)$$

where

$$M_{11}(s) = P^T R_{(\mathcal{F}, \mathcal{C})}^T (sI + L_{\text{ess}}(\mathcal{F}))^{-1} L_e(\mathcal{F}) R_{(\mathcal{F}, \mathcal{C})} P$$

$$M_{12}(s) = P^T R_{(\mathcal{F}, \mathcal{C})}^T (sI + L_{\text{ess}}(\mathcal{F}))^{-1} E(\mathcal{F})^T$$

$$M_{21}(s) = E(\mathcal{G}_o)^T (E_{\mathcal{F}}^L)^T (sI + L_{\text{ess}}(\mathcal{F}))^{-1} L_e(\mathcal{F}) R_{(\mathcal{F}, \mathcal{C})} P$$

$$M_{22}(s) = E(\mathcal{G}_o)^T (E_{\mathcal{F}}^L)^T (sI + L_{\text{ess}}(\mathcal{F}))^{-1} E(\mathcal{F})^T.$$

This representation can lead directly to a *small-gain* interpretation for the allowable edge-weight uncertainties that guarantees the system is robustly stable. In particular, a sufficient condition for determining whether $(I - M_{11}\Delta)$ has a stable proper inverse is to ensure that $\|M_{11}(s)\Delta\|_\infty < 1$.

We now cite a result from [13] that gives insight on the \mathcal{H}_∞ norm of the transfer function matrix $M_{11}(s)$.

Proposition V.1 ([13]):

$$\|M_{11}(s)\|_\infty = \bar{\sigma}(M_{11}(0)).$$

This results shows that the \mathcal{H}_∞ performance of the system $M_{11}(s)$ can be obtained by computing the largest singular value of the real matrix $M_{11}(0)$, leading to the result on the robust stability of $\Sigma_{\mathcal{F}}(\mathcal{G}, \Delta)$.

Theorem V.2: Let $\Delta = \{\Delta \in \mathbb{R}^{|\mathcal{E}_\Delta| \times |\mathcal{E}_\Delta|}, \|\Delta\| \leq \bar{\delta}, \Delta \text{ diagonal}\}$ and assume $\Sigma_{\mathcal{F}}(\mathcal{G})$ is nominally stable. Then, the uncertain edge agreement protocol is robustly stable for any $\Delta \in \Delta$ if $\|\Delta\| < (\sigma(M_{11}(0)))^{-1}$.

Theorem V.2 is, in fact, a direct statement of the small-gain theorem and can be considered conservative. One might wonder if the results of Theorem IV.3 might lead to a less conservative result. That is, if \mathcal{E}_Δ forms a cut set of \mathcal{G} , is it possible that $\max_{e \in \mathcal{E}_\Delta} \mathcal{W}(e) < \bar{\sigma}(M_{11}(0))^{-1}$. The following result shows that this cannot be the case.

Proposition V.3:

$$\left(\max_{e \in \mathcal{E}_\Delta} \mathcal{W}(e)\right)^{-1} \leq \max_{e \in \mathcal{E}_\Delta} \mathcal{R}_e(\mathcal{G}) \leq \bar{\sigma}(M_{11}(0)) \leq \mathbf{R}_{\text{tot}}^{\mathcal{E}_\Delta}$$

where

$$\mathbf{R}_{\text{tot}}^{\mathcal{E}_\Delta} = \text{trace} \left[P^T E^T (E_{\mathcal{T}}^L)^T \left(R_{(\mathcal{T}, \mathcal{C})}^T W R_{(\mathcal{T}, \mathcal{C})}^T \right)^{-1} E_{\mathcal{T}}^L E P \right].$$

Proof: Let $E_\Delta = EP$ denote the incidence matrix with columns corresponding to the edges in \mathcal{E}_Δ . Observe that $R_{(\mathcal{F}, \mathcal{C})}P = E_{\mathcal{T}}^L E_\Delta$. Therefore, the diagonal entries of $M_{11}(0)$ correspond to the effective resistance between nodes incident to the edges in \mathcal{E}_Δ [from (12)], establishing a lower bound on $\bar{\sigma}(M_{11}(0))$. By a similar argument, it follows that $\text{trace}[M_{11}(0)] = \mathbf{R}_{\text{tot}}^{\mathcal{E}_\Delta}$ leading to the upper bound.

To establish the lower bound, let $S, S' \subset \mathcal{V}$ be the set of nodes incident to edges in \mathcal{E}_Δ (with S the nodes that are the head of each edge, and S' the tail). For each edge $e \in \mathcal{E}_\Delta$, let $r_e = \mathcal{W}(e)^{-1} = w_e$ denote its resistance. Consider now the edges $e_1 = (u, v), e_2 = (u', v') \in \mathcal{E}_\Delta$ and assume $r_1 \geq r_2$ (equivalently, $w_1 \leq w_2$). Similarly, let $\tilde{\mathcal{R}}_{uv} = \mathcal{R}_{uv}(\mathcal{G} \setminus e_1)$ denote the effective resistance between u and v after edge e_1 is removed. It now follows that:

$$\mathcal{R}_{uv}(\mathcal{G}) = \frac{r_1 \tilde{\mathcal{R}}_{uv}}{r_1 + \tilde{\mathcal{R}}_{uv}} \geq \frac{r_2(r_1 + \tilde{\mathcal{R}}_{uv} - r_1)}{r_1 + \tilde{\mathcal{R}}_{uv}} \geq r_2.$$

Since the above inequality holds for any pair of edges in \mathcal{E}_Δ and, in particular, when $r_2 = (\max_{e \in \mathcal{E}_\Delta} \mathcal{W}(e))^{-1}$, the lower bound is verified. ■

By introducing additional assumptions into the system $\Sigma_{\mathcal{F}}(\mathcal{G}, \Delta)$, the small gain result of Theorem V.2 can be tightened and endowed with more meaningful physical interpretations.

Theorem V.4: Consider the uncertain edge agreement protocol $\Sigma_{\mathcal{F}}(\mathcal{G}, \Delta)$ with $\mathcal{E}_\Delta = \mathcal{E}$ and the nominal edge weights are all positive and equal, that is, $W = \alpha I$ with $\alpha > 0$. Then, $\|M_{11}(s)\|_\infty = \alpha^{-1}$ and $\Sigma_{\mathcal{F}}(\mathcal{G}, \Delta)$ is robustly stable for all diagonal Δ , satisfying $\|\Delta\|_\infty = \bar{\sigma}(\Delta) < \alpha$.

Proof: With $\mathcal{E}_\Delta = \mathcal{E}$, it follows that $P = I$. Therefore $M_{11}(0) = (1/\alpha)R_{(\mathcal{F}, \mathcal{C})}^T(R_{(\mathcal{F}, \mathcal{C})}R_{(\mathcal{F}, \mathcal{C})}^T)^{-1}R_{(\mathcal{F}, \mathcal{C})}$ is a projection matrix scaled by α^{-1} , that is, $\|M_{11}(s)\|_\infty = \bar{\sigma}(M_{11}(0)) = \alpha^{-1}$. The bound on the uncertainty Δ then follows immediately from the small-gain theorem. ■

While still a conservative result, Theorem V.4 has a more intuitive interpretation. It states that all edge weights must remain positive even in the presence of uncertainty. This result can also be understood in the context of Theorem IV.3 since one can imagine the uncertainty attacking a set of edges that forms a cut in the graph. Thus, if the uncertainty is such that the edges in the cut have negative weight (i.e., $w_k + \delta_k < 0$), then the resulting edge agreement protocol will be unstable.

The small gain result can, in fact, lead to an exact condition for the robust stability of $\Sigma_{\mathcal{F}}(\mathcal{G}, \Delta)$. The next result considers the uncertain edge agreement problem when the uncertainty is present in only a single edge in the graph.

Theorem V.5: Consider the uncertain edge agreement protocol $\Sigma_{\mathcal{F}}(\mathcal{G}, \Delta)$ with $\mathcal{E}_\Delta = \{\{u, v\}\}$ (i.e., $|\mathcal{E}_\Delta| = 1$) and assume $\Sigma(\mathcal{G})$ is nominally stable. Then, $\|M_{11}(s)\|_\infty = \mathcal{R}_{uv}(\mathcal{G})$ and $\Sigma(\mathcal{G}, \Delta)$ are robustly stable for all δ satisfying $\|\Delta\|_\infty < \mathcal{R}_{uv}^{-1}(\mathcal{G})$.

Proof: This statement directly follows Theorem IV.7, Proposition V.1, and the effective resistance between two nodes in a graph (11). ■

Theorem V.5 provides an exact condition for the robust stability of $\Sigma_{\mathcal{F}}(\mathcal{G}, \Delta)$ when a single edge has uncertainty, that is, there is no conservatism in this result, which is a direct consequence of Theorem IV.7. This result also demonstrates that the effective resistance between nodes in the network can serve the role of a *gain margin* for the system.

Consider now a scenario where an attacker is able to perturb the weight of any single edge in the graph. From Theorem V.5, each edge can tolerate a perturbation determined by the effective resistance between the two incident nodes. Thus, an attacker might consider choosing the edge with the “smallest” margin. This leads to a less restrictive uncertainty class than in Theorem V.5; however, the result will consequently be more conservative.

Corollary V.6: Let $\Delta = \{\delta P_i P_i^T, i=1, \dots, |\mathcal{E}|, |\delta| \leq \bar{\delta}\}$ be the set of allowable perturbations on the edge weights, where $P_i \in \mathbb{R}^{|\mathcal{E}|}$ satisfies $[P_i]_j = 1$ when $j=i$ and 0 otherwise, and assume that $\Sigma_{\mathcal{F}}(\mathcal{G})$ is nominally stable. Then, $\Sigma(\mathcal{G}, \Delta)$ is robustly stable if for any $\Delta \in \Delta$, one has $\|\Delta\|_\infty < \min_{uv} \mathcal{R}_{uv}^{-1}(\mathcal{G})$.

Corollary V-6 demonstrates the earlier discussion. If only a single edge can be “attacked,” then the robustness margin of the entire network will be determined by the edge connecting the nodes with the largest effective resistance between them. It should be noted that this value will depend on the nominal values of the edge weights along with the location of the edges in the graph.

Using the result of Theorem IV.8 also leads to a less conservative corollary for Theorem V.2.

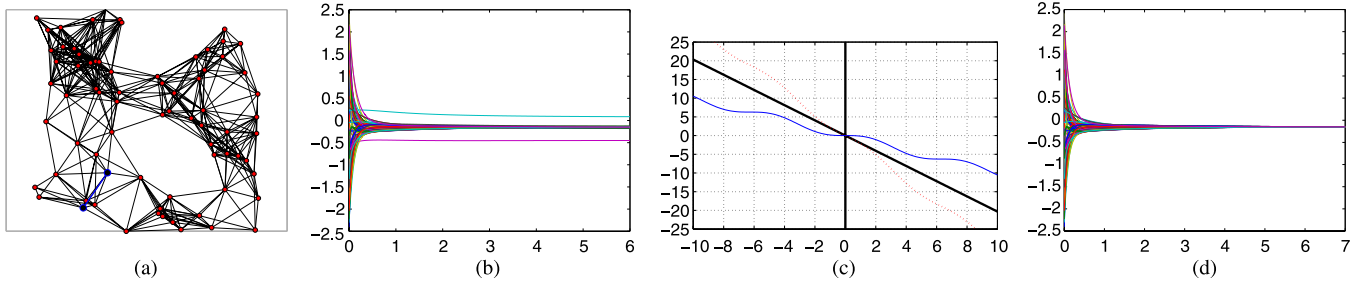


Fig. 3. Simulation results demonstrating Corollary V.6 and V.11. (a) Random geometric graph. (b) Clustering trajectories. (c) Nonlinear coupling functions. (d) Corollary V.11 is satisfied.

Corollary V.7: Let \mathcal{E}_Δ denote the set of uncertain edges and define the uncertainty set as $\mathbf{\Delta} = \{\Delta \in \mathbb{R}^{|\mathcal{E}_\Delta| \times |\mathcal{E}_\Delta|}, \|\Delta\| \leq \bar{\delta}, \Delta \text{diagonal}\}$. For each edge $k = (u, v) \in \mathcal{E}_\Delta$, define $\mathcal{P}_k \subseteq \mathcal{E}$ as the set of all edges in \mathcal{G} that belong to a path connecting nodes u to v and assume that $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$ for all $i, j \in \mathcal{E}_\Delta$ with $\mathbf{R} = \text{diag}\{\mathcal{R}_1(\mathcal{G}), \dots, \mathcal{R}_{|\mathcal{E}_\Delta|}(\mathcal{G})\}$. Assume that $\Sigma_{\mathcal{F}}(\mathcal{G})$ is nominally stable. Then, $\Sigma(\mathcal{G}, \Delta)$ is robustly stable for all $\Delta \in \mathbf{\Delta}$, if $\|\Delta\| < \min_{e \in \mathcal{E}_\Delta} \mathcal{R}_e(\mathcal{G})^{-1}$.

B. Robust Stability of $\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi)$

We now examine the robust stability of the consensus system with nonlinear couplings $\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi)$ given in (6). First, we recall results on the multivariable circle criterion [27].

Let $K_1, K_2 \in \mathbb{R}^{|\mathcal{E}_\Delta| \times |\mathcal{E}_\Delta|}$ be real diagonal matrices with $K_2 - K_1 > 0$. Define Φ to be the set of all sector-bounded time-varying nonlinearities as described in Section III-A with $K_1 = \text{diag}\{\alpha_1, \dots, \alpha_{|\mathcal{E}_\Delta|}\}$ and $K_2 = \text{diag}\{\beta_1, \dots, \beta_{|\mathcal{E}_\Delta|}\}$. In the following text, we denote by $G(s)$ the transfer-function of a linear system with state-space matrices (A, B, C, D) .

Lemma V.8 ([27]): The following statements are equivalent:

- 1) The state matrix A is asymptotically stable and $G(s)$ is strictly positive real.
- 2) $D + D^T > 0$ and there exists $X, Y > 0$ such that

$$0 = A^T X + X A + (B^T X - C)^T (D + D^T)^{-1} (B^T X - C) + Y.$$

Theorem V.9 ([27]): If $(I + K_2 G(s))(I + K_1 G(s))^{-1}$ is strictly positive real, then for all $\Phi \in \Phi$, the negative feedback interconnection of $G(s)$ and Φ is asymptotically stable.

We are now prepared to make a general statement on the robust stability of $\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi)$.

Theorem V.10: Consider the nonlinear edge agreement protocol $\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi)$ with $\mathcal{E}_\Delta \subseteq \mathcal{E}$ and assume $\Sigma(\mathcal{G})$ is nominally stable. Then, $\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi)$ is asymptotically stable for any $\Phi \in \Phi$ if

$$\|K_1\| = \max_i |\alpha_i| < \frac{1}{\|M_{11}(s)\|_\infty}$$

and $2W + P(K - (1 - \sqrt{2})I)(K - (1 + \sqrt{2})I)P^T > 0$, where $K = K_2 - K_1$.

Proof: From Lemma v.8, it follows that a necessary condition for $(I + K_2 M_{11}(s))(I + K_1 M_{11}(s))^{-1}$ to be strictly positive real is for $(I + K_1 M_{11}(s))^{-1}$ to have a stable and proper inverse. Since K_1 may have negative entries, we arrive at the small-gain result on the largest singular value K_1 to guarantee

the inverse exists. Next, it can be shown that a minimal state-space realization of the transfer function $(I + K_2 M_{11}(s))(I + K_1 M_{11}(s))^{-1}$ can be expressed by the state-space matrices

$$\begin{aligned} \tilde{A} &= -L_{\text{ess}}(\mathcal{F}) - L_e(\mathcal{F})R_{(\mathcal{F}, C)}PK_1P^TR_{(\mathcal{F}, C)}^T \\ \tilde{B} &= L_e(\mathcal{F})R_{(\mathcal{F}, C)}P, \tilde{C} = (K_2 - K_1)P^TR_{(\mathcal{F}, C)}^T \\ \tilde{D} &= I. \end{aligned}$$

Using Theorem V.9, it can be verified that choosing $X = L_e(\mathcal{F})^{-1}$ and $Y = R_{(\mathcal{F}, C)}QR_{(\mathcal{F}, C)}^T$ with

$$Q = 2W - PP^T - P(K_2 - K_1)^2P + 2P(K_2 - K_1)P^T$$

satisfies the equality stated in condition 2). Thus, $Y > 0$ if and only if $Q > 0$ concluding the proof. ■

As expected, the agreement protocol with sector nonlinearities on the edges will inherit certain robustness measures found in the case where there are additive perturbations. We can understand the aforementioned result better when considering the case where $|\mathcal{E}_\Delta| = 1$; that is, the nonlinearity is present on only a single edge.

Corollary V.11: Consider the nonlinear edge agreement protocol $\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi)$ with $\mathcal{E}_\Delta = \{\{u, v\}\}$ (i.e., $|\mathcal{E}_\Delta| = 1$) and assume $\Sigma(\mathcal{G})$ is nominally stable. Then, $\Sigma_{\mathcal{F}}(\mathcal{G}, \Phi)$ is asymptotically stable for all $\Phi \in \Phi$, satisfying

$$|\alpha| < \mathcal{R}_{uv}^{-1}(\mathcal{G}) \text{ and } ((\beta - \alpha)^2 - 2(\beta - \alpha) - 1) > -2w_{uv}.$$

The result of Corollary V.11 can also be interpreted in terms of passivity indices quantifying the shortage or excess of passivity in the system. Indeed, Corollary V.11 can be used to conclude that the effective resistance between a pair of nodes with nonlinear couplings represents an *excess of passivity* for the system.

VI. SIMULATION EXAMPLE

We now briefly report on a numerical simulation illustrating the main results of this work. Fig. 3(a) shows a random geometric graph with 75 nodes. The edge weights are taken to be the inverse of the Euclidean distance between neighboring nodes. With $\mathcal{E}_\Delta = \mathcal{E}$, Corollary V.6 states that the robustness margin corresponds to the inverse of the largest effective resistance between nodes in \mathcal{E}_Δ which, in this example, corresponds to the edge connecting the two black nodes in Fig. 3(a); the effective resistance between these nodes is computed to be 0.429.

Fig. 3(b) shows the resulting trajectories when the uncertainty on that edge is exactly equal to the inverse of the effective resistance, leading to a *clustering phenomena* (considered in detail in [15]).

Using the same graph, Figs. 3(c) shows two nonlinear coupling functions used across the uncertain edge highlighted earlier ($y = ax + \sin x$). Choosing the nonlinearity satisfying the sector condition from Corollary V.11 ($a < -2.035$) leads to stable trajectories [Fig. 3(d)], while choosing $a = -3$ [red line in Fig. 3(c)] leads to unstable dynamics.

VII. CONCLUDING REMARKS

This paper examined the robust stability of weighted consensus protocols with bounded additive uncertainties on the edge weights. The main results demonstrate that the robustness margins of such systems are determined by the combinatorial properties of the uncertain edges, and the nominal value of the edge weights. These margins were related to the effective resistance in a network, and the robust stability results were cast in this framework. We believe that this framework provides a new graph-theoretic interpretation for classical notions from robust control theory when applied to networked systems. The ability for this framework to also handle nonlinear extensions suggests a greater utility and our future works aim to apply these results to certain real-world applications.

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