On Sampled-Data Consensus: Divide and Concur

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***Abstract*—This note studies the consensus problem for integrator agents under intermittent information exchange between connected neighbours at asynchronous sampling time instances. It proposes a novel sampled-data protocol, based on *emulating* suitable global analog consensus dynamics at each agent and using sampled *centroids* of these emulators to convey information between agents. We show that the closed-loop dynamics can be divided into centroid and disagreement parts. The former is completely autonomous and evolves according to time-varying discrete consensus dynamics, independent of the sampling intervals. The disagreement part evolves according to conventional analog consensus dynamics for a constant network topology and is driven by the emulator centroids. The system then asymptotically converges to agreement under mild assumptions on the persistency of connectivity and the uniform boundedness of sampling intervals. A substantially simplified and scalable implementation under a special emulated topology, namely the complete graph, is also proposed.**

***Index Terms*—Sampled-data systems, network control systems.**

I. Introduction and Problem Setup

Reaching an agreement between autonomous dynamical systems in the presence of communication constraints is a fundamental problem in many applications, both in control and in other areas [1]–[3]. In its arguably simplest form, the problem can be formulated in continuous time for a group of ν first-order autonomous agents described by

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{0i}, \tag{1}$$

for all $i \in \mathbb{N}_\nu$ (all integers in $[1, \nu]$). We say that the agents are in *agreement* if,

$$x_i(t) = x_j(t), \quad \forall i, j \in \mathbb{N}_\nu. \tag{2}$$

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So the agreement problem is to choose control inputs u_i that lead dynamics (1) to agreement asymptotically under bounded x_i and u_i. This would be an elementary problem if every agent had continuous access to the states of all other agents. However, this scenario is not practical in networked applications, where information exchange is costly. Hence, additional constraints are introduced.

A. Spatial Constraints

A natural assumption in large-scale networks is that each agent can interact only with its "close" neighbours. Formally, this can be expressed as the constraint that the ith agent can communicate only with agents whose indices belong to a fixed *neighborhood set* $\mathcal{N}_i \subset \mathbb{N}_\nu \setminus \{i\}$, whose cardinality is $|\mathcal{N}_i| < \nu$.

The agreement problem for (1) is then known to be solved by the celebrated consensus protocol [1]–[3],

$$u_i(t) = -\kappa \left(|\mathcal{N}_i| x_i(t) - \sum_{j \in \mathcal{N}_i} x_j(t) \right), \tag{3}$$

for some $\kappa > 0$ (for the sake of simplicity we work with uniform weights on all the edges). Here, the form of the control (3) emphasizes that agents exchange state-information by *communication* (as opposed to relative sensing, which leads to the diffusively coupled form of the consensus protocol $u_i = \kappa \sum_{j \in \mathcal{N}_i} (x_j - x_i)$). That is, all agents in the neighborhood set of agent i continuously, instantaneously, and reliably broadcast their state value to agent i.

The analysis of the resulting closed-loop system is particularly elegant in the aggregate form, through the use of graph theory. Namely, we can model the spatial constraints using an undirected graph with Laplacian matrix $L = D - A$, where $D = \operatorname{diag}\{|\mathcal{N}_i|\}$ is the degree matrix, and A is the adjacency matrix, with $a_{ij} = 1$ if $j \in \mathcal{N}_i$ and 0 otherwise. The aggregate version of the consensus protocol then reads

$$u(t) = -\kappa L x(t), \text{ where } \left[\begin{array}{c:c} x(t) & u(t) \end{array}\right] := \left[\begin{array}{c:c} x_1(t) & u_1(t) \\ \vdots & \vdots \\ x_\nu(t) & u_\nu(t) \end{array}\right],$$

and results in the collective closed-loop dynamics

$$\dot{x}(t) = -\kappa L x(t), \quad x(0) = x_0. \tag{4}$$

If the corresponding graph is connected, then L has a single eigenvalue at the origin (its right eigenvector is $\mathbb{1}$) and all other eigenvalues are strictly positive. Consequently, the closed-loop system has its only equilibrium in the *agreement set* satisfying (2). If the connection topology is symmetric, meaning $j \in \mathcal{N}_i \iff i \in \mathcal{N}_j$, the dynamics above converge to the *average agreement* at $(1/\nu)\mathbb{1}'x_0$ exponentially, where $\mathbb{1}$ is the all-ones vector. The convergence rate is determined by the second smallest eigenvalue of κL and can be made arbitrarily fast by increasing the gain κ, see [3, Sec. 3.1].

B. Temporal Constraints

Networked implementations may impose additional limitations on information exchange between agents, this time related to time instances at which it is possible. Specifically, we assume that agents can convey information about their states to neighbours only at time instances $t = s_k$, $k \in \mathbb{Z}_+$, for a strict monotonically increasing sequence of *sampling instances* $\{s_k\}_k$. If all agents transmit their states simultaneously, at each s_k, sampling is said to be *synchronous*. If the ith agent at each s_k receives information from only a subset of its neighbors, sampling is referred to as *asynchronous*. This formalism can also naturally accommodate changing connection topologies.

There is rich literature on sampled-data consensus, studying problems under synchronous and asynchronous sampling, time- and event-triggered sampling mechanisms, see [4] and the references therein. Still, the vast majority of available approaches assumes that the control signal is piecewise constant, unchanged between updates from neighbours, i.e., uses the zero-order hold as the D/A converter. This assumption facilitates the reduction of the problem to a pure discrete agreement, at least under synchronous sampling. Indeed, in the latter case the discretized aggregate system is

$$x[k+1] = x[k] + h_k u[k],$$

where $x[k] := x(s_k)$ and $h_k := s_{k+1} - s_k$ (sampling interval). A version of the discrete consensus protocol,

$$u[k] = -\kappa L x[k],$$

leads to the closed-loop dynamics

$$x[k+1] = (I - \kappa h_k L)x[k], \tag{5}$$

which can be thought of as the Euler approximation of (4). System (5) still has its only equilibrium at an agreement set, yet its stability is now guaranteed only if κh_k is sufficiently small for all possible sampling intervals h_k [1]. This compromises the convergence rate for networks with large variability in sampling intervals. There are other agreement-reaching protocols for discrete systems [1], but they all require sufficiently small, and normally conservative, gains to guarantee stability. Even more cautious gains would be required under asynchronous sampling [5].

Remark 1 (Discrete vs. Sampled-Data): The simple case considered above exemplifies well a key difference between pure discrete-time and sampled-data networked setups. The latter typically has shift-varying discretized models, as realistic network sampling is intermittent. Moreover, parameters of those models are uncertain, unless we somehow know the next sampling instance.

C. Proposed Setup and Contribution

This note argues for parting with the use of the zero-order hold, synchronized with measurement updates, in sampled-data consensus protocols. This custom does not appear to be justified by implementation requirements nowadays and might be an atavism, survived from the early days of computer-controlled systems. We aim at exploiting opportunities offered by the use of more sophisticated hold mechanisms to reach agreement over agents (1) under both spatial (e.g., near-neighbor only) and temporal (sampling) constraints.

The design of control waveforms, or D/A converters, is not new to sampled-data control per se. In the centralized setting, optimal converters for periodic [6]–[8] and intermittent [9] sampling rates can be designed. Similar control waveform generators are also used in the networked control context in [10], [11], also without spatial constraints. Their common property is that they in effect *emulate* the control signal of the desired analog closed-loop control in an open-loop way. This property was observed in [8, Sec. 6] and conjectured as a guiding principle for situations where direct performance-justified design of the hold function is not available.

We follow that logic to put forward a sampled-data consensus protocol based on emulating (3). The challenge is that such an emulation has to be carried out in a distributed manner, locally at each agent. We work it out by implementing a model of the *whole world*, viz. (4), at each agent in continuous time. Each local controller uses then the states of these emulators to mimic the continuous-time consensus protocol (3). Information about real states of neighbouring agents is used to adjust local emulators each time this information becomes available at sampling instances.

We propose sample and update protocols, based on the *centroids* of the corresponding emulators of each agent. With such a choice, the resulting closed-loop dynamics can be divided into those of the decoupled centroids, which behave as an autonomous system, and those of local disagreements, which evolves according to the analog consensus dynamics driven by the emulator centroids. Remarkably, the dynamics of the centroids are also independent of the sampling intervals, i.e., are certain, unlike (5). As such, the analysis is simplified and global asymptotic convergence to agreement is proved under mild connectivity assumptions on the sampled-data topology and without any need in *a priori* knowledge of sampling instances and specific bounds on sampling intervals. Another noteworthy property of the proposed architecture is that the spatial topology emulated locally need not match that of the actual network. This property is exploited to substantially reduce computational complexity, rendering it independent of the number of agents.

This letter is organized as follows. In Section II we formalize the proposed architecture. The stability and agreement in the resulting closed-loop dynamics are then analyzed in Section III. Section III-A discusses a special choice of local topologies, which simplifies the implementation. In Section IV an example illustrating the proposed algorithm is presented. We wind up with concluding remarks in Section V.

Notation: The sets of all non-negative integers is denoted as $\mathbb{Z}_+$ and $\mathbb{N}_\nu := \{i \in \mathbb{Z} \mid 1 \leq i \leq \nu\}$. Given a set $\mathcal{S} \subset \mathbb{Z}$, its cardinality is denoted as $|\mathcal{S}|$. Sequences with indices from $\mathbb{Z}_+$ are indicated as $\{s_i\}_i$.

By e_i we understand the ith standard basis vector in $\mathbb{R}^\nu$, by $\mathbb{1}_\nu$, or simply $\mathbb{1}$ when the dimension is clear from the context, the all-ones vector from $\mathbb{R}^\nu$, and by I_ν, or simply I, the $\nu\times\nu$ identity matrix. The complex-conjugate transpose of a matrix M is denoted by M'. The notation $\operatorname{diag}\{M_i\}$ stands for a block-diagonal matrix with diagonal elements M_i. The image (range) and kernel (null) spaces of a matrix M are notated $\operatorname{Im} M$ and $\ker M$, respectively. The orthogonal projection onto $\operatorname{Im}\mathbb{1}_\nu$ is $P_{\mathbb{1}} := \mathbb{1}_\nu\mathbb{1}_\nu'/\nu$.

II. Proposed Sampled-Data Architecture

As discussed in the Introduction, we aim at reaching agreement for the group of ν agents described by (1) under given spatial and temporal communication constraints. These constraints are determined by a strict monotonically increasing sequence of sampling instances $\{s_k\}_k$ and associated sets $\mathcal{N}_i[k]\subset\mathbb{N}_\nu\setminus\{i\}$ of neighbors of the ith agent that convey information about their states at each s_k. These information updates are intrinsically directional and not necessarily symmetric, i.e., $j\in\mathcal{N}_i[k]$ might not imply that $i\in\mathcal{N}_j[k]$. Each set $\mathcal{N}_i[k]$ can thus be associated with a directed graph, say $\mathcal{G}[k]$, having the Laplacian $L[k]\in\mathbb{R}^{\nu\times\nu}$. To ensure persistent connectivity in the whole scheme, we assume hereafter that

$\mathcal{A}_1$: there is a strictly increasing sub-sequence of sampling indices $\{k_n\}_n$ such that for all $n\in\mathbb{Z}_+$ (i) the intervals $s_{k_{n+1}}-s_{k_n}$ are uniformly bounded and (ii) $\bigcup_{k=k_n+1}^{k_{n+1}}\mathcal{G}[k]$ contains a directed rooted tree.

Assumption $\mathcal{A}_1$ is commonly employed in works related to coordination protocols over switching or time-varying graphs [2], [12], [13]. This assumption ensures that information propagates throughout the entire network persistently across bounded sampling intervals, leaving no nodes forever detached from the rest of the network.

Following the discussion in Section I-C, the proposed architecture is based on emulating the "ideal" analog consensus behavior à la (4) at each agent. To this end, associate with each agent an emulated spatial communication topology, which is represented by a graph $\hat{\mathcal{G}}$ with Laplacian $\hat{L}\in\mathbb{R}^{\nu\times\nu}$, and the corresponding set of neighbors $\hat{\mathcal{N}}_i\subset\mathbb{N}_\nu\setminus\{i\}$ of the ith agent. We emphasize that this $\hat{\mathcal{G}}$ need not match the actual spatio-temporal topology represented by $\mathcal{G}[k]$. The emulated topology is constrained to be the same for each agent with the assumption that

$\mathcal{A}_2$: the graph $\hat{\mathcal{G}}$ is undirected and connected, i.e., that $\hat{L}=\hat{L}'$ and its eigenvalue at the origin is single.

Remark 2 (Undirected vs. Directed Graphs): The emulator topology may also be directed. We require primarily the property that $\hat{L}$ has a single eigenvalue at the origin with corresponding left eigenvector $\mathbb{1}$. This can be also achieved, for example, by weakly connected and balanced directed graphs.

By the emulator at the ith agent we then understand the function $\mu_i(t)\in\mathbb{R}^\nu$, whose elements $\mu_{ij} := [\mu_i]_j$ satisfy

$$\begin{aligned}\dot{\mu}_{ij}(t) &= -\kappa\sum_{l\in\hat{\mathcal{N}}_j}\bigl(\mu_{ij}(t)-\mu_{il}(t)\bigr),\quad \forall j\in\mathbb{N}_\nu\setminus\{i\}\\ \mu_{ii}(t) &= x_i(t),\end{aligned}\tag{6}$$

for some given initial conditions $\mu_{ij}(0)$. It is readily seen that (6) at each $j\neq i$ matches the jth row of (4). The purpose of (6) is to emulate the analog consensus protocol (3) at the ith agent by replacing the actual (remote) neighbouring states x_j by their local clones μ_{ij}, i.e., as

$$u_i(t) = -\kappa\left(|\hat{\mathcal{N}}_i|\mu_{ii}(t)-\sum_{j\in\hat{\mathcal{N}}_i}\mu_{ij}(t)\right).\tag{7}$$

Taking into account that $\dot{\mu}_{ii}=u$, by (1), we end up with

$$\dot{\mu}_i(t) = -\kappa\hat{L}\mu_i\tag{8}$$

as the collective dynamics of the ith agent.

It should be clear that the control law (6)–(7) is incomplete, for the resulted dynamics (8) are autonomous, not synchronized with other agents. We thus need to complement it by a synchronization mechanism satisfying given spatio-temporal constraints.

To this end, two aspects are to be decided: (i) what information agents should broadcast about their own states, and (ii) how emulators should utilize the conveyed information. One can think of a number of possible approaches here, even if each agent may broadcast only a scalar signal. For instance, agents may broadcast their own sampled states, $x_j(s_k)$, or a function of the state of their complete emulator, $\phi(\mu_j(s_k))$. The receiving-side emulators may then update only components corresponding to the received updates, e.g., as $\mu_{ij}(s_k^+)=x_j(s_k)$, or all their states simultaneously.

Our choice, motivated mainly by analysis benefits, is to broadcast the *centroid* of the corresponding emulator,

$$\bar{\mu}_j(t) := \frac{1}{\nu}\mathbb{1}'\mu_j(t) = \frac{1}{\nu}\sum_{j=1}^{\nu}\mu_{ij}(t),$$

and to update components of μ_i at the receiver end as

$$\mu_{ij}(s_k^+) = \mu_{ij}(s_k)-\alpha_{ij}\sum_{l\in\mathcal{N}_i[k]}(\bar{\mu}_i(s_k)-\bar{\mu}_l(s_k))\tag{9}$$

for all $j\neq i$ and some gains $\alpha_{ij}\in\mathbb{R}$. Because $\mu_{ii}=x_i$, the ith component of μ_i can only be affected via u_i and is thus not updated at sampling instances. Summing up these updates and dividing them by ν, we have the following update algorithm for the centroid of the ith emulator:

$$\bar{\mu}_i(s_k^+) = \bar{\mu}_i(s_k)-\sum_{j\neq i}\frac{\alpha_{ij}}{\nu}\sum_{l\in\mathcal{N}_i[k]}(\bar{\mu}_i(s_k)-\bar{\mu}_l(s_k)),$$

which bears a resemblance to a discrete consensus protocol.

III. Closed-Loop System and Its Agreement

Having determined the control architecture, we are now in the position to analyze the closed-loop system and its capability to reach agreement. The first step in that direction is, naturally, to characterize the closed-loop dynamics.

Lemma 1: Consider the set of agents described by (1) and controlled by (7), where components of the emulators satisfy (6) and (9). The cumulative state of the resulted closed-loop system $\mu(t)\in\mathbb{R}^{\nu^2}$ satisfies the hybrid equations,

$$\begin{cases}\dot{\mu}(t) = -\kappa(I_\nu\otimes\hat{L})\mu(t), & \mu(0)=\mu_0\\ \mu(s_k^+) = A_{\mathrm{d}}[k]\mu(s_k),\end{cases}\tag{10}$$

for initial conditions μ_0 comprised those of all μ_{ij}, where

$$A_{\rm d}[k] := I_{\nu^2} - \frac{1}{\nu}\sum_{i=1}^{\nu}\sum_{j\in\mathcal{N}_i[k]}\big(e_i(e_i - e_j)'\big)\otimes(\alpha_i\mathbb{1}'), \quad (11)$$

and $\alpha_i = \begin{bmatrix}\alpha_{i1} & \cdots & \alpha_{i\nu}\end{bmatrix}'$ satisfy $e_i'\alpha_i = 0$ for all $i \in \mathbb{N}_\nu$.

Proof: The "flow" part of (10) is the cumulative block-diagonal version of (8). The "jump" part follows from (9) by the relations $\mu = \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} e_i \otimes e_j \mu_{ij}$, $\bar{\mu}_i = (e_i' \otimes \mathbb{1}')\mu/\nu$, and with a little help of the mixed-product property of the Kronecker product [14, Th. 13.3]. ■

Equation (10) represents hybrid dynamics, with the continuous flow part, whose evolution is shaped by the block-diagonal matrix $-\kappa(I_\nu \otimes \hat{L})$, and the discontinuous jump part, whose evolution is shaped by $A_{\rm d}[k]$. The former matrix is actually the Laplacian of the union of ν disconnected clones of $\hat{\mathcal{G}}$. These clones are then connected at jump stages. The result below plays a key role to understand properties of this connectivity mechanism.

Lemma 2: Every $A_{\rm d}[k]$ defined by (11) satisfies

$$A_{\rm d}[k](I_\nu \otimes (I_\nu - P_{\mathbb{1}})) = I_\nu \otimes (I_\nu - P_{\mathbb{1}}), \quad (12a)$$

where $I_\nu - P_{\mathbb{1}}$ is the orthogonal projection onto $\ker \mathbb{1}'$. Also,

$$(I_\nu \otimes \mathbb{1}')A_{\rm d}[k] = \left(I_\nu - \frac{1}{\nu}L[k]\right)\otimes \mathbb{1}', \quad (12b)$$

whenever $\mathbb{1}'\alpha_i = 1$ for all $i \in \mathbb{N}_\nu$.

Proof: By the mixed-product property of the Kronecker product, $M_{\rm r}[k] := (I_{\nu^2} - A_{\rm d}[k])(I \otimes (I - P_{\mathbb{1}}))$ satisfies

$$M_{\rm r}[k] = \frac{1}{\nu}\sum_{i=1}^{\nu}\sum_{j\in\mathcal{N}_i[k]}\big(e_i(e_i - e_j)'\big)\otimes(\alpha_i\mathbb{1}'(I - P_{\mathbb{1}})) = 0,$$

which proves (12a). Likewise,

$$\begin{aligned} M_{\rm l}[k] &:= (I_\nu \otimes \mathbb{1}')(I_{\nu^2} - A_{\rm d}[k]) \\ &= \frac{1}{\nu}\sum_{i=1}^{\nu}\sum_{j\in\mathcal{N}_i[k]}\big(e_i(e_i - e_j)'\big)\otimes(\mathbb{1}'\alpha_i\mathbb{1}') \\ &= \frac{1}{\nu}\sum_{i=1}^{\nu}\sum_{j\in\mathcal{N}_i[k]}\big(e_i(e_i - e_j)'\big)\otimes\mathbb{1}' = \frac{1}{\nu}L[k]\otimes\mathbb{1}', \end{aligned}$$

which yields (12b). ■

It follows from (12b) that every $A_{\rm d}[k]$ has at least $\nu(\nu-1)$ Eigenvalues at 1 and their right Eigenspace, $\operatorname{Im}[I \otimes (I - P_{\mathbb{1}})]$, is independent of k. Hence, jumps alone cannot result in an agreement between agents either. But, as we show below, alternating flow and jump propagation actions does lead to agreement in the system.

To show that, we need two signals,

$$\bar{\mu}(t) := \frac{1}{\nu}(I_\nu \otimes \mathbb{1}')\mu(t), \quad \delta(t) := (I_\nu \otimes (I_\nu - P_{\mathbb{1}}))\mu(t),$$

which are the ν-dimensional vector comprised of the centroids $\bar{\mu}_i(t)$, and the ν^2-dimensional vector of local disagreements, respectively. It is readily seen that

$$\mu(t) = (I \otimes \mathbb{1})\bar{\mu}(t) + \delta(t),$$

and $\|\mu(t)\|^2 = \nu\|\bar{\mu}(t)\|^2 + \|\delta(t)\|^2$ owing to the orthogonality of the centroid and disagreement. Hence, the boundedness of both $\bar{\mu}$ and δ implies that of μ itself. Moreover, because

$$x(t) = \sum_{i=1}^{\nu} e_i(e_i' \otimes e_i')\mu(t) = \bar{\mu}(t) + \sum_{i=1}^{\nu} e_i(e_i' \otimes e_i')\delta(t),$$

the states of agents x_i agree whenever the centroids agree, i.e., $\bar{\mu}(t) \in \operatorname{Im}\mathbb{1}$, and local disagreements vanish, i.e., $\delta(t) \to 0$. In this direction, we now show that for an appropriate choice of the weight vector α_i, the emulator centroid dynamics evolve according to a discrete consensus protocol over a switching graph.

Lemma 3: If $\mathbb{1}'\alpha_i = 1$ for all $i \in \mathbb{N}_\nu$, then

$$\bar{\mu}(s_{k+1}) = \left(I - \frac{1}{\nu}L[k]\right)\bar{\mu}(s_k) \quad (13)$$

at sampling instances s_k, and is constant between samples, i.e., at times $t \in (s_k, s_{k+1}]$.

Proof: From the jump equation in (10), we can analyze the emulator centroid dynamics as

$$\bar{\mu}(s_k^+) = \frac{1}{\nu}(I_\nu \otimes \mathbb{1}')A_{\rm d}[k]\mu(s_k) = \left(I_\nu - \frac{1}{\nu}L[k]\right)\bar{\mu}(s_k),$$

where (12b) was used for the second equality. Similarly, it follows from the flow equation in (10) that the centroid dynamics are invariant, i.e.,

$$\dot{\bar{\mu}}(t) = \frac{1}{\nu}(I_\nu \otimes \mathbb{1}')\dot{\mu}(t) = -\frac{\kappa}{\nu}(I_\nu \otimes (\mathbb{1}'\hat{L}))\mu(t) = 0,$$

because $\mathbb{1}'\hat{L} = 0$. ■

The result of Lemma 3 says that the dynamics of the centroids are completely decoupled from the rest of the state of (10), i.e., δ, and are driven only by the interaction topology at sampling instances. This is reminiscent of standard discrete consensus dynamics, like (5). An important difference from the latter is that the parameters of (13) do not depend on the sampling intervals.

Having established that the emulator centroid dynamics are decoupled from those of the disagreement vector, we now focus our attention on analyzing the disagreement dynamics.

Lemma 4: If $\mathbb{1}'\alpha_i = 1$ for all $i \in \mathbb{N}_\nu$, then

$$\begin{cases} \dot{\delta}(t) = -\kappa(I_\nu \otimes \hat{L})\delta(t), \quad \delta(0) = \delta_0 \\ \delta(s_k^+) = \delta(s_k) + B_{\rm d}[k]L[k]\bar{\mu}(s_k), \end{cases} \quad (14)$$

where $\delta_0 := (I_\nu \otimes (I_\nu - P_{\mathbb{1}}))\mu_0$ satisfies $(I \otimes \mathbb{1}')\delta_0 = 0$ and

$$B_{\rm d}[k] := \sum_{i=1}^{\nu}(e_i e_i') \otimes (\mathbb{1}/\nu - \alpha_i)$$

satisfies $(I \otimes \mathbb{1}')B_{\rm d}[k] = 0$.

Proof: By $\mathcal{A}_\mathbf{2}$, $\mathbb{1}'\hat{L} = 0$, so $(I - P_{\mathbb{1}})\hat{L} = \hat{L}(I - P_{\mathbb{1}})$, whence the flow part of (14) follows directly from that of (10). The jump part of (10) leads then to

$$\begin{aligned} \delta(s_k^+) &= (I \otimes (I - P_{\mathbb{1}}))A_{\rm d}(\delta(s_k) + (I \otimes \mathbb{1})\bar{\mu}(s_k)) \\ &= \delta(s_k) + (I \otimes (I - P_{\mathbb{1}}))A_{\rm d}(I \otimes \mathbb{1})\bar{\mu}(s_k), \end{aligned}$$

which can be derived by the fact that $(I \otimes (I - P_{\mathbb{1}}))\delta = \delta$ and (12a). By (12b), we have that

$$(I \otimes (I - P_{\mathbb{1}}))A_{\rm d}(I \otimes \mathbb{1}) = (A_{\rm d} - I)(I \otimes \mathbb{1}) + L[k] \otimes \mathbb{1}/\nu,$$

and then by (11), that

$$\begin{aligned}(I - A_{\mathrm{d}})(I \otimes \mathbb{1}) &= -\sum_{i=1}^{\nu} \sum_{j \in \mathcal{N}_i[k]} \bigl(e_i(e_i - e_j)'\bigr) \otimes \alpha_i \\ &= -\sum_{i=1}^{\nu} (e_i e_i' L[k]) \otimes \alpha_i \\ &= -\left(\sum_{i=1}^{\nu} (e_i e_i') \otimes \alpha_i\right) L[k].\end{aligned}$$

Because $L[k] \otimes \mathbb{1} = (I \otimes \mathbb{1})L[k] = (\sum_i (e_i e_i') \otimes \mathbb{1})L[k]$, we have $(I \otimes (I - P_{\mathbb{1}}))A_{\mathrm{d}}(I \otimes \mathbb{1}) = B_{\mathrm{d}}[k]L[k]$ and end up with the jump part of (14). ■

Lemma 4 says that δ, similarly to μ, satisfies a hybrid dynamic equation. The difference of the equation for δ, (14), from that for μ, (10), is that the former has constant "A" matrices not only in its flow part, but also for jumps. The only varying part is the discrete "B" matrix, through which the exogenous input $\bar{\mu}$ affects δ. This matrix does not affect the stability of the system, so the stability and convergence analyses are greatly simplified. In fact, if $L[k]\bar{\mu}(s_k) = 0$, then (14) has no jumps and comprises effectively ν clones of the continuous-time consensus dynamics (4), except that $\delta(t)$ is kept orthogonal to $\operatorname{Im} \mathbb{1}$ for all t. The latter is one of key properties leading to the main result of this letter.

Theorem 1: If $\mathcal{A}_{\mathbf{1,2}}$ hold, then agents (1) controlled by (7) with emulators (6), (9) converge asymptotically to $\operatorname{Im} \mathbb{1}$ for all initial conditions, all sampling sequences with uniformly bounded sampling intervals $s_{k+1} - s_k$, and all emulator update gains α_{ij} such that $\sum_{j \neq i} \alpha_{ij} = 1$, for all $i \in \mathbb{N}_\nu$. Moreover, the emulators remain bounded and agree asymptotically as well.

Proof: We need to show that the vector of centroids, $\bar{\mu}$, agrees and the the vector of local disagreements, δ, vanishes asymptotically. So consider $\bar{\mu}$ first. By $\mathcal{A}_{\mathbf{1}}$, system (13) satisfies the conditions of [2, Lemma 2.29, Th. 2.37] and thus $\bar{\mu}(t)$ is bounded and converges to $\operatorname{Im} \mathbb{1}_\nu$. This also implies that the sequence $\{L[k]\bar{\mu}(s_k)\}_k$ vanishes asymptotically.

Now, move to δ. The first result that we need is the stability of the autonomous version of (14), under $L[k]\bar{\mu}(s_k) = 0$ for all $k \in \mathbb{Z}_+$. This is a standard result for continuous-time disagreement dynamics. Namely, it is known [1, Sec. II-D] that $\|\delta(t)\|$ is bounded and vanishes exponentially, with the rate determined by the smallest nonzero eigenvalue of $\hat{L}$, which is the algebraic connectivity of $\hat{\mathcal{G}}$. Thus, $\delta(t)$ is the state of an exponentially stable linear system, whose exogenous input is bounded and asymptotically vanishing. Hence, $\delta \to 0$ and is bounded. ■

A. The Choice of the Complete Graph as $\hat{\mathcal{G}}$

An obvious problem with implementing emulators is that their dimension equals the number of agents. Emulating all agents might not be feasible for large-scale networks. Yet this problem can be resolved by an appropriate choice of the emulated connectivity graph $\hat{\mathcal{G}}$, which is in our power.

To this end, note that each agent broadcasts only the centroid of its emulator, which actually does not change between updates. The only obstacle preventing then to emulate only the centroid is the need in individual components of μ_i in the control law (7). But if the emulated graph is the complete graph, whose $\hat{L} = \nu(I_\nu - P_{\mathbb{1}})$ and $\hat{\mathcal{N}}_i = \mathbb{N}_\nu \setminus \{i\}$, (7) reads

$$u_i(t) = -\kappa\left(\nu\mu_{ii}(t) - \sum_{j=1}^{\nu} \mu_{ij}(t)\right) = -\kappa\nu(x_i(t) - \bar{\mu}_i(t)).$$

Hence, with this choice we do not need individual μ_{ij} to implement u_i either. This, in turn, allows to drop explicit emulators. The control law becomes then

$$u_i(t) = -\kappa\nu\bigl(x_i(t) - \bar{\mu}_i(s_k^+)\bigr), \quad \forall t \in (s_k, s_{k+1}], \tag{7$'$}$$

and the emulator updates (9) reduce to the updates of their centroids according to

$$\bar{\mu}_i(s_k^+) = \frac{1}{\nu}\left((\nu - |\mathcal{N}_i[k]|)\bar{\mu}_i(s_{k-1}^+) + \sum_{j \in \mathcal{N}_i[k]} \bar{\mu}_j(s_k)\right), \tag{9$'$}$$

where the condition $\sum_j \alpha_{ij} = 1$, required in Theorem 1, is used. Note that the control signal in (7$'$) is still not piecewise constant, as the local feedback is analog.

The controller defined by (7$'$) and (9$'$) has an intuitive interpretation. Namely, (7$'$) is the proportional analog servo system for agent (1) with the piecewise-constant $\bar{\mu}_i$ as its reference signal. This reference is then updated according to the discrete consensus protocol (9$'$) with reference signals of neighboring agents. This is a reasonable strategy in the case when local agents are easy to control, but the information about the outside world is hard to acquire. This logic appears to extend seamlessly to the cases of higher-order agents, unmeasurable states, and disturbances.

Remark 3 (Impulsive Hold): If we were allowed to use the impulsive control signals, we could cause the actual state of every agent to jump at each sampling time instance. In this case the choice $\alpha_i = \mathbb{1}/\nu$ would be possible. With this choice $B_{\mathrm{d}}[k] = 0$ for all k, rendering δ in (14) completely decoupled from $\bar{\mu}$. In that scenario, the selection of agreeing initial conditions for each emulator, i.e., $\mu_i(0) \in \operatorname{Im} \mathbb{1}$, would keep each emulator in agreement for all t and allow us to implement only centroids of each emulator again, now for every $\hat{\mathcal{G}}$ satisfying $\mathcal{A}_{\mathbf{2}}$.

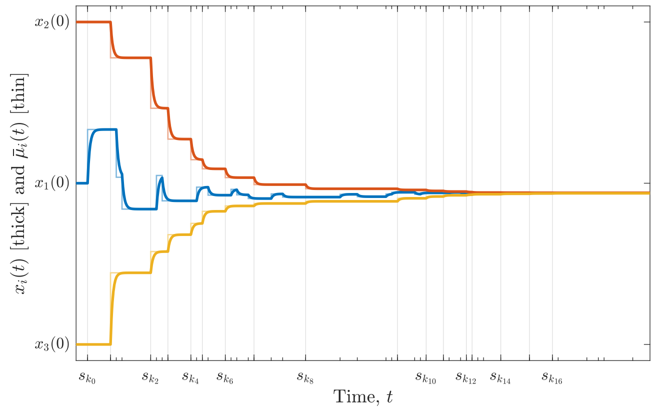
IV. Illustrative Example

To illustrate the proposed sampled-data protocol, consider the simple system comprised of $\nu = 3$ agents under the fixed (spatial) interaction topology

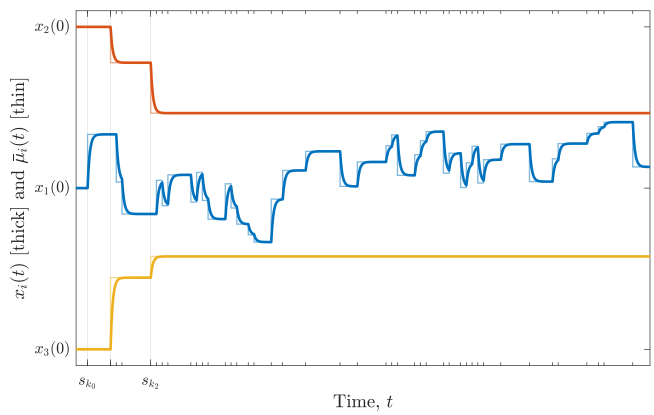
$$\mathcal{G} = \{(1, 2), (1, 3), (2, 1), (3, 1)\},$$

and asynchronous intermittent communication. The edge (i, j) indicates that the ith agent conveys its centroid to the jth one. By the asynchronous communication we mean that each agent transmits only at a subset of sampling instances. Consequently, in this example each $\mathcal{G}[k]$ may be the union of any nonempty subset of the graphs $\{(1, 2), (1, 3)\}$, $\{(2, 1)\}$, and $\{(3, 1)\}$.

It should be clear that $\cup_k \mathcal{G}[k]$ contains a directed rooted tree if and only if it contains $\{(1, 2), (1, 3)\}$. In other words, the first agent serves as a fulcrum, facilitating information



(a) $\mathcal{A}_1$ holds



(b) $\mathcal{A}_1$ does not hold

Fig. 1. Agents x_i (thick lines) and centroids $\bar{\mu}_i$ (thin lines).

exchange between the other agents. Hence, $\mathcal{A}_1$ holds if and only if the first agent transmits persistently, with uniformly bounded intervals. The sequence $\{k_n\}_n$ in $\mathcal{A}_1$ may then comprise all indices of sampling instances, at which the first agent transmits.

Fig. 1 presents simulation results in the interval $t \in [0, 30]$, where the trajectories of the agents, $x_i(t)$, are depicted by thick lines and those of the centroids, $\bar{\mu}_i(t)$, are represented by thin lines. Sampling instances, shown by abscissa ticks, are a random variable such that $s_{k+1} - s_k \in 0.3\,\mathbb{N}_7$. Major ticks indicate the sub-sequence of sampling instances $\{k_n\}_n$ defined in $\mathcal{A}_1$. The emulators use the complete graph with $\hat{L} = 3(I_3 - P_{\mathbb{1}})$, as described in Section III-A, with $\kappa = 3$.

First, we simulate the system for which $\mathcal{A}_1$ holds true. Specifically, the transmitting agent at each sampling instance s_k is a random pick from the set $\{1, 2, 3\}$. We can see from the plots in Fig. 1(a) that the trajectories of the agents exhibit the behaviour discussed at the end of Section III-A, namely those of simple first order systems tracking piecewise-constant reference signals. Because agents are modeled as simple integrators, there is no local steady-state error. An increase (decrease) of κ would accelerate (slow down) local tracking. In any case, the centroids expectably converge to an agreement point, leading x_i to satisfy (2) asymptotically.

The situation is different when the first agent stops transmitting its information. Assume that this happens for the previous simulation after $t = s_{k_2}$, when the set of transmitting agents reduces to $\{2, 3\}$. The result shown in Fig. 1(b) demonstrates that even multi-consensus, in which agents converge to a finite number of clusters, might not be reachable then. In our case the failure of the first agent to transmit creates, in a sense, a tug of war between the second and the third agents. The first agent gets stuck in the middle and keeps oscillating.

V. Concluding Remarks

In this note we have put forward a novel approach to solving the sampled-data consensus problem under intermittent and asynchronous sampling. The proposed architecture yields global asymptotic agreement under very mild connectivity assumptions. It was further shown that a particular choice of analog architecture can greatly reduce the complexity of the overall controller, resulting in a simple servo loop with a piecewise constant reference signal. Analysis of the controller's performance under (9) as well as other updating protocols is currently being investigated. Furthermore, the relatively weak assumptions required to guarantee convergence hint at potential synergy with event-triggering mechanisms.

In a broader perspective, the inherent separation between control and information processing offered by the proposed approach is particularly appealing. The methods introduced were derived for the consensus problem, but can potentially be extended to more general dynamics and multi-agent control goals. Such extensions are subject to current research.

References

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[2] W. Ren and R. W. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control: Theory and Applications*. London, U.K.: Springer-Verlag, 2008.

[3] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton, NJ, USA: Princeton Univ. Press, 2010.

[4] X. Ge, Q.-L. Han, D. Ding, X.-M. Zhang, and B. Ning, "A survey on recent advances in distributed sampled-data cooperative control of multi-agent systems," *Neurocomputing*, vol. 275, pp. 1684–1701, Jan. 2018.

[5] M. Mattioni, "On multiconsensus of multi-agent systems under aperiodic and asynchronous sampling," *IEEE Control Syst. Lett.*, vol. 4, no. 4, pp. 839–844, Oct. 2020.

[6] G. Tadmor, "H_∞ optimal sampled-data control in continuous time systems," *Int. J. Control*, vol. 56, no. 1, pp. 99–141, 1992.

[7] W. Sun, K. M. Nagpal, and P. P. Khargonekar, "$\mathcal{H}_\infty$ control and filtering for sampled-data systems," *IEEE Trans. Autom. Control*, vol. 38, no. 8, pp. 1162–1174, Aug. 1993.

[8] L. Mirkin, H. Rotstein, and Z. J. Palmor, "H^2 and H^∞ design of sampled-data systems using lifting. Part I: General framework and solutions," *SIAM J. Control Optim.*, vol. 38, no. 1, pp. 175–196, 1999.

[9] L. Mirkin, "Intermittent redesign of analog controllers via the Youla parameter," *IEEE Trans. Autom. Control*, vol. 62, no. 4, pp. 1838–1851, Apr. 2017.

[10] L. A. Montestruque and P. J. Antsaklis, "On the model-based control of networked systems," *Automatica*, vol. 39, no. 10, pp. 1837–1843, 2003.

[11] J. Lunze and D. Lehmann, "A state-feedback approach to event-based control," *Automatica*, vol. 46, no. 1, pp. 211–215, 2010.

[12] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[13] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 169–182, Feb. 2005.

[14] A. Laub, *Matrix Analysis for Scientists & Engineers*. Philadelphia, PA, USA: SIAM, 2005.