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Robust Consensus of Higher Order Agents over Cycle Graphs

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This paper considers identical agents modeled as higher order linear systems whose dynamics are not precisely known and obtains conditions for consensus of a system of such imprecisely modeled identical linear systems. The state information is exchanged between the agents over a cycle graph. The imprecision in the models of the agents, represented by gain margins of certain transfer functions, is directly related to the parameters of the cycle digraph and it is shown that the number of agents can play a significant role in ensuring consensus among the agents. Additionally, perturbations are also considered in the edge weights of the cycle graph and the tolerable limits on such perturbations are derived. Thus, combined effects of these two forms of uncertainties/perturbations are investigated in this work. Simulations support the theoretical results.

I. Introduction

Robustness of consensus over weighted undirected [1] and directed graphs [2] have been studied, where perturbations were considered in the edge weights. Most of these works, however, considered single or double integrator models of the agents [3, 4] for consensus. Some have studied higher order integrators [5] and yet others have considered higher order linear models of the agents [6–8]. In [9], robustness of such higher order consensus systems were studied by considering additive uncertainties in the agent model while [10] considered linear parameter varying agent models. Thus, robustness studies of the consensus protocol have proceeded along two directions. On the one hand, parameters of the network, such as edge weights are considered to be uncertain or perturbed, while agent models are precisely known. Along the other direction, the uncertainty in the multi-agent system is incorporated in the form of uncertainties in the agent models while the network topology is robust. In this work, an attempt is made to tackle both of these uncertainties simultaneously and to investigate the bearing that one form of uncertainty has on the other. The uncertainty in the dynamics of the n identical agents is incorporated by considering each agent to be a member of a family of interval plants.

While the directed cycle graph, which describes cyclic pursuit, has been of interest to researchers in multi-agent systems [11–16], the undirected cycle graph also has some important roles to play in the performance of consensus seeking systems, as shown in [3,17]. Therefore, as a starting point for studying the combined effects of edge weight perturbations and model uncertainties in agents, the cycle graph provides a nice framework. In this paper, the communication topology among agents is, therefore, chosen as a cycle, C_n

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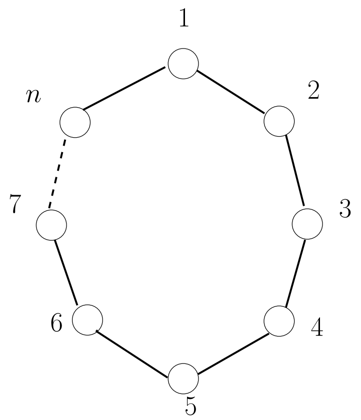

Figure 1: A cycle graph over n vertices, C_n.

as shown in Fig. 1. Every agent has exactly two neighbors and the number of edges in the graph is equal to the number of agents.

The main contributions of this paper are as follows. First, unlike most of the literature dealing with consensus, this paper considers higher order linear models of agents that are not necessarily integrators. Secondly, though the agents are identical, it is assumed that their models are not precisely known. Under this condition, it is desired to design a network, with suitable edge weights, for exchange of state information among agents, that leads to consensus among the agents' states. Whether such a network can be designed depends on the precision with which the model of the agent is known. In other words, the question before the designer may be summed thus. Given that a designed network leads to consensus of a certain number of identical linear agents, modeled by general higher order linear systems, if all the agents are now replaced by another set of identical agents having the same order of dynamics as before but different physical parameters, will consensus result? If the answer is a 'yes' then the network is *robust* to perturbations in the agent model.

Thirdly, this paper also considers the designed edge weights to be perturbed from their nominal value. The question then becomes one of obtaining the amount of perturbation that the edge weights can tolerate before at least some member of the family of uncertain agent models loses consensus. Unlike the case of integrators, where positive definiteness of the Laplacian matrix, or lack thereof, sufficed to answer the question of allowable edge weight perturbations, in this case the lack of precisely known agent models leads to more stringent requirements on the Laplacian spectra.

The paper is organized in the following manner. Section II revisits some concepts of algebraic graph theory that are useful in analyzing networked dynamic systems and also reviews the notions of interval plants that help model uncertain linear systems. The main theoretical results are stated and proved in Section III. In Section IV, simulations vindicate these theoretical results. Finally, Section V concludes the paper with some scope for future studies.

II. Mathematical Preliminaries

A. Algebraic Graph Theory

This section reviews some concepts related to weighted graphs, and some commonly used notations [18]. A weighted undirected graph $\mathcal{G}$ comprises a set of vertices, $\mathcal{V}$, a set of edges

$\mathcal{E}$, which is an unordered pair of distinct vertices, and a diagonal matrix $W \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{E}|}$. The diagonal entry, w_{kk}, of W is the weight on an edge $e_k \in \mathcal{E}$. Throughout this work $|\mathcal{V}| = n = |\mathcal{E}|$. Vertices i and j are neighbors if $\{i, j\} \in \mathcal{E}$ and vertex j (or vertex i) is said to belong to $\mathcal{N}_i$ (correspondingly, $\mathcal{N}_j$), the neighbor set for vertex i (or j). An *orientation* is assigned to the graph, $\mathcal{G}$, such that every edge has a direction. This means that there is an initial and terminal vertex corresponding to every edge, e_k, joining vertices i and j. This then becomes an ordered pair (i, j). The matrix $E(\mathcal{G})$ is the incidence matrix, a $\{0, \pm 1\}$ matrix with rows and columns indexed by the vertices and edges of $\mathcal{G}$, respectively, such that

$$[E(\mathcal{G})]_{ij} = \begin{cases} +1 \text{ if } i \text{ is the initial node of edge } e_k \\ -1 \text{ if } i \text{ is the terminal node of edge } e_k \\ 0 \text{ otherwise.} \end{cases}$$

The graph and edge Laplacian matrices are defined in terms of the incidence matrix, $E(\mathcal{G})$, and the weight matrix, W, as $\mathcal{L} = E(\mathcal{G})WE(\mathcal{G})^T$, the graph Laplacian, [18] and $\mathcal{L}_e = E(\mathcal{G})^T E(\mathcal{G})W$, the edge Laplacian [3].

B. Interval Plants

The characteristic polynomial of a precisely known linear system has constant real coefficients. But, if the physical parameters of the plant are not precisely known, the coefficients of the characteristic polynomial may vary within some real interval. By the same token, the numerator polynomial of the transfer function of such a system may have coefficients belonging to different real intervals. Such polynomial are called *interval polynomials* and they are of the form $\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \ldots + \delta_n s^n$, where the coefficients lie in the range $\delta_i \in [x_i, y_i] \subset \mathbb{R}$, $i = 1, 2, \ldots, n$. These interval polynomials gives rise to *interval plants*, represented by transfer functions that are the ratios of two such interval polynomials [19, 20].

A key result pertaining to the stability of interval polynomials is Kharitonov's Theorem, stated below.

Theorem 1. *(Kharitonov's Theorem [19, 20]) Suppose $\mathcal{I}(s)$ is a set of real polynomials of degree n given by $\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \ldots + \delta_n s^n$, where the coefficients lie in the range $\delta_i \in [x_i, y_i]$, $i = 1, 2, \ldots, n$. Every polynomial in the family $\mathcal{I}(s)$ is Hurwitz if and only if the following four extreme polynomials are Hurwitz:*

$$\begin{aligned} K_1(s) &= x_0 + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4 + x_5 s^5 + y_6 s^6 + \cdots \\ K_2(s) &= x_0 + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4 + y_5 s^5 + y_6 s^6 + \cdots \\ K_3(s) &= y_0 + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4 + y_5 s^5 + x_6 s^6 + \cdots \\ K_4(s) &= y_0 + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4 + x_5 s^5 + x_6 s^6 + \cdots \end{aligned}$$

For a given $s = j\omega^*$, the set of values of the polynomials in $\mathcal{I}(j\omega^*)$ form a rectangle in the complex plane, shown in Fig. 2, with the set $\{K_i(j\omega^*)\}_{i=1,\ldots,4}$ being the vertices of the rectangle. This rectangle is known as the *value set.*

III. Main Results

A network of dynamical systems, modeled as interval plants, is considered here. The agents exchange their full-state information over a cycle graph C_n. Determining condi-

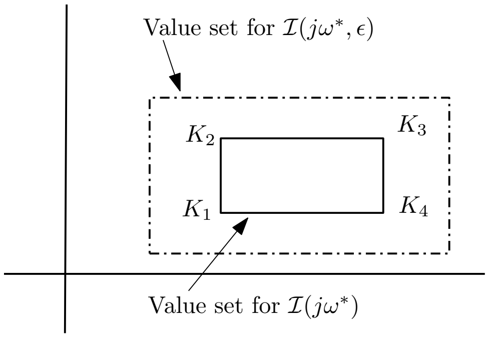

Figure 2: Value sets at a fixed ω^* for $\mathcal{I}(s)$ and $\mathcal{I}(s, \epsilon)$.

tions that ensure full-state consensus of the network is the aim of this study. In other words, for every pair of agents i, j, it is required that

$$\lim_{t\to\infty} \|x_i^{(k)}(t) - x_j^{(k)}(t)\| = 0,\ k = 0, \ldots, m-1,$$

where $x_i^{(k)}(t)$ is the k-th time derivative of the state $x_i(t)$.

All the agents are considered to have identical dynamics. However, their dynamics are not known precisely giving rise to a family of interval plants that are used to model each agent as follows:

$$x_i^{(m)} + \alpha_{m-1} x_i^{(m-1)} + \ldots + \alpha_0 x_i = u_i. \tag{1}$$

where the coefficients α_j, $j = 0, 1, \ldots, m-1$ are not accurately known, but belong to an interval on the real axis, that is, $\alpha_j \in [\underline{\alpha}_j, \bar{\alpha}_j] \subset \mathbb{R}\ \forall j$.

The following assumption is sometimes made in this work to ensure that the states are bounded at consensus, as in [6]. If boundedness is not required, then this assumption may be removed.

Assumption 1. *There exists an $\epsilon > 0$, such that the family of interval polynomials $\mathcal{I}(s, \epsilon) := s^m + [\underline{\alpha}_{m-1} - \epsilon, \bar{\alpha}_{m-1} + \epsilon]s^{m-1} + \ldots + [\underline{\alpha}_0 - \epsilon, \bar{\alpha}_0 + \epsilon]$, where $\mathcal{I}(s, 0)$ contains the characteristic polynomial of the n identical agents, is Hurwitz.*

Remark 1. *Assumption 1 is not too restrictive. Even if the exact models of the dynamic systems, represented by the nodes of a graph, are not precisely known, it is not too restrictive to assume that their dynamics are nevertheless stable. The ϵ-refinement only adds a certain margin of stability for the entire uncertain family as illustrated by the value set for a given frequency ω^* in Fig. 2.*

A. Uncertain Higher Order Agents: Consensus Design

The main problem addressed in this subsection may be stated below.

Problem 1. *For the set of n identical agents, exchanging state information over C_n, with uncertain dynamics, modeled as interval plants, design a linear consensus protocol by choosing positive edge weights on C_n.*

The goal is to obtain a linear consensus protocol for agents modeled by higher order interval plants, similar to the ones for single and double integrators [3, 4]. Similar to [6], it is desired that all the agents converge to the consensus state, $\xi(t)$, which evolves along $\xi^{(m)} + \alpha_{m-1}\xi^{(m-1)} + \ldots + \alpha_0 \xi = 0$ with $\alpha_j \in [\underline{\alpha}_j, \bar{\alpha}_j]\ \forall j$.

The control law for agent i is chosen as

$$u_i = \sum_{l=0}^{m-1} \beta_l[(w_{i,i+1}(x_{i+1}^{(l)} - x_i^{(l)})) + (w_{i,i-1}(x_{i-1}^{(l)} - x_i^{(l)}))], \tag{2}$$

where $w_{i,i+1}$ is the weight on the edge connecting nodes i and $i+1$. The problem thus becomes one of choosing the parameters $\{\beta_l\}_{l=0,\ldots,m-1}$ and the set of n edge weights. In this work, $w_{i,i+1} = \mu, \forall i$ (i is modulo n) will be considered

Let the position states of the agents be stacked as $x = [x_1 \ldots x_p \ldots x_n]^T$. Subsequently, these states and their higher derivatives may be stacked together as follows:

$$\boldsymbol{x} = \begin{bmatrix} x^T & \cdots & x^{(l)T} & \cdots & x^{(m-1)T} \end{bmatrix}^T. \tag{3}$$

Now, plugging in the control law in (2), the system can be described as

$$\dot{\boldsymbol{x}} = \bar{\boldsymbol{A}}\boldsymbol{x} \tag{4}$$

$$\bar{\boldsymbol{A}} = \begin{bmatrix} 0 & \boldsymbol{I}_n & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{I}_n \\ \boldsymbol{\Lambda}_0 & \boldsymbol{\Lambda}_1 & \cdots & \boldsymbol{\Lambda}_{m-1} \end{bmatrix}, \tag{5}$$

where $\boldsymbol{\Lambda_j} = -\alpha_j \boldsymbol{I}_n - \beta_j \mathcal{L}$ and $\mathcal{L}$ is the weighted graph Laplacian of C_n.

The characteristic equation for the system in (4) is:

$$P(s) = \det[s^m \boldsymbol{I}_n + \sum_{j=0}^{m-1} (\alpha_j \boldsymbol{I}_n + \beta_j \mathcal{L}) s^j] = 0. \tag{6}$$

From Lemma 4 in [6] it follows that the characteristic polynomial, $P(s)$, in (6) is

$$P(s) = \Pi_{i=1}^n [s^m + \sum_{j=0}^{m-1} (\alpha_j + \beta_j \lambda_i(\mathcal{L})) s^j], \tag{7}$$

where $\lambda_i(\mathcal{L})$ is the i-th eigenvalue of the graph Laplacian $\mathcal{L}$. From [6], it further follows that, for consensus, the polynomial $\bar{P}(s)$, given by

$$\bar{P}(s) = \Pi_{i=2}^n [s^m + \sum_{j=0}^{m-1} (\alpha_j + \beta_j \lambda_i(\mathcal{L})) s^j], \tag{8}$$

where $\lambda_1(\mathcal{L}) = 0$, needs to be Hurwitz. However, Assumption 1 is only required to ensure that the states are bounded at consensus.

To analyze and interpret the requirements on the Laplacian spectrum, for consensus, it is convenient to consider a related standard SISO system, such as the one in Fig. 3. Consider the family of transfer functions (interval plants) given by

$$G(s) = \frac{\beta_{m-1}s^{m-1} + \beta_{m-2}s^{m-2} + \ldots + \beta_0}{s^m + \alpha_{m-1}s^{m-1} + \ldots + \alpha_0}, \tag{9}$$

where $s^m + \alpha_{m-1}s^{m-1} + \ldots + \alpha_0$ is an interval polynomial and $\beta_{m-1}s^{m-1} + \beta_{m-2}s^{m-2} + \ldots + \beta_0$ is a polynomial with constant coefficients, chosen by the designer or is fixed for the system.

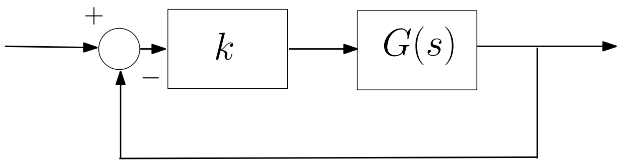

Figure 3: Equivalent interval plant in closed loop with proportional control.

Remark 2. *Due to Assumption 1, there always exists a choice of the set* $\{\beta_l\}_{l=0,\ldots,m-1}$ *such that the closed loop system in Fig. 3 is stable for* $k = 1$.

If Assumption 1 is lifted, then the uncertain dynamics of the identical agents may not be stable. In such a case, the designer has to carefully choose the set $\{\beta_l\}_{l=0,\ldots,m-1}$ to robustly stabilize the entire family of uncertain dynamical systems given by (1) in closed loop. The problem now becomes one of obtaining a range of positive real values for k such that the family of interval plants is robustly stabilized in closed loop. Since the coefficients, α_i, of the agents' dynamical equation vary independently, the solution to this problem is equivalent to the application of Kharitonov's theorem on the interval polynomial $K(s)$ given by

$$\begin{aligned} K(s) = s^m &+ [\underline{\alpha}_{m-1} + k\beta_{m-1}, \bar{\alpha}_{m-1} + k\beta_{m-1}]s^{m-1} \\ &+ [\underline{\alpha}_{m-2} + k\beta_{m-2}, \bar{\alpha}_{m-2} + k\beta_{m-2}]s^{m-2} + \ldots \\ &+ [\underline{\alpha}_0 + k\beta_0, \bar{\alpha}_0 + k\beta_0]. \end{aligned} \tag{10}$$

Remark 2 guarantees a choice of $\{\beta_l\}_{l=0,\ldots,m-1}$ and proportional controller for the plant $G(s)$ in Fig. 3 if Assumption 1 holds. However, in some cases the designer may not have the freedom to choose $\{\beta_l\}_{l=0,\ldots,m-1}$ freely, so that some of the open loop zeros of $G(s)$ in (9) may lie on the open right half plane even though the poles are in the left half plane. In such cases, the system will have an upper gain margin, $\bar{k}$. But, it is not difficult to choose edge weights μ that ensures the Laplacian spectral radius is bounded above by $\bar{k}$.

Moreover, if Assumption 1 does not hold, then there is also a lower bound on the gain k, say $\underline{k}$, such that for any $k < \underline{k}$, the closed loop system will be unstable. Once again, there will always exist a choice of μ that ensures the non-zero eigenvalues of the Laplacian are bounded below by $\bar{k}$.

Suppose, without loss of generality, the application of Kharitonov's theorem on (10) specifies a range of real values of k, the proportional gain, such that for all $k \in (\underline{k}, \bar{k})$, the closed loop system in Fig. 3 is robustly stable. The requirement for consensus may now be specified by the following lemma.

Lemma 1. *If all the non-zero eigenvalues of the Laplcian* $\mathcal{L}$ *belong to the interval* $(\underline{k}, \bar{k})$, *then the multi-agent system in* (1) *with control law* (2), *achieves consensus in its states, where the family of transfer functions* $G(s)$ *in* (9) *are robustly stable for* $k \in (\underline{k}, \bar{k})$, *for a particular choice of the set* $\{\beta_l\}_{l=0,\ldots,m-1}$.

Proof. Comparing eqns. (7) and (9), it is apparent that if the gain k in Fig. 3 is chosen to be an eigenvalue of the Laplacian $\mathcal{L}$, then the closed loop poles of (9) are also zeros of the polynomial (7). Now, if all the nonzero eigenvalues of $\mathcal{L}$ belong to the interval $(\underline{k}, \bar{k})$, then the zeros of $P(s)$ in (7), except the roots of the agent's characteristic equation, will all be in the left half of the complex plane. This is because the open loop transfer function, $G(s)$, which corresponds to the zero eigenvalue of $\mathcal{L}$ is robustly stable by Assumption 1. Applying Theorem 3 of [6] completes the proof of consensus. □

Corollary 1. *In case there are multiple disjoint intervals $\mathcal{P}_s = (\underline{k}_s, \bar{k}_s)$ such that the closed loop interval plant in Fig. 3 is stable for $k \in \mathcal{P}_s$, for consensus it suffices to ensure that the non-zero eigenvalues of $\mathcal{L}$ belong to $\cup_s \mathcal{P}_s$.*

Proof. Suppose the non-zero eigenvalues of $\mathcal{L}$ belong to $\cup_s \mathcal{P}_s$. Clearly, every polynomial of the form $s^m + \sum_{j=0}^{m-1}(\alpha_j + \beta_j \lambda_i(\mathcal{L}))s^j$ is Hurwitz, where λ_i is a non-zero eigenvalue of $\mathcal{L}$. Thus, the polynomial $P(s)$ in (7) is also Hurwitz, leading to consensus. $\square$

The following result holds for any graph Laplacian matrix.

Lemma 2. *For any positive number d and a weighted undirected graph $\mathcal{G}$, there always exists a set of positive edge weights that ensures $||\mathcal{L}||_2 < d$.*

Proof. The graph Laplacian for a weighted undirected graph $\mathcal{G}$ is written as EWE^T and its norm is given by $||\mathcal{L}||_2 = \max_{||x||_2=1} ||EWE^T x|| \leq \max_{||x||_2=1} ||EW||||E^T x||_2 \leq ||W||||E||^2$. Suppose $\max_{||x||_2=1} ||E^T x||_2 = D$. Clearly, choosing $W = \mu \boldsymbol{I}_q$ (q being the number of edges, $|\mathcal{E}|$), where $0 < \mu < d/D^2$, ensures $||\mathcal{L}||_2 < d$. $\square$

Remark 3. *If $\underline{k} < 0$, then for consensus it is sufficient to choose edge weights to ensure $||\mathcal{L}||_2 < \bar{k}$, where $\bar{k} > 0$. According to Lemma 2, such a choice of edge weights always exists. However, when $\underline{k} > 0$, then the edge weights must be chosen such that the non-zero eigenvalues of $\mathcal{L}$ are bounded above and below by positive real numbers.*

Remark 4. *If Assumption 1 is removed so that some of the poles of* (9) *are in the rhp, then $\underline{k} > 0$ holds, provided the zeros are in the lhp. In such cases, the choice of $\{\beta_l\}_{l=0,\dots,m-1}$ is crucial as this will ensure that there exists some $k > 0$, such that the interval family will be closed loop stable.*

The main result of this subsection may now be stated.

Theorem 2. *The multi-agent system in* (1) *with control law* (2) *can achieve consensus in its states, for a given choice of $\{\beta_l\}_{l=0,\dots,m-1}$, if the following inequality holds:*

$$\frac{\underline{k}}{\bar{k}} < 4\gamma \sin^2(\pi/2n) \tag{11}$$

where any $k \in [\underline{k}, \bar{k}]$ ($\underline{k} > 0$) ensures the robust closed loop stability of the interval plant in (9)*, and $\gamma = 1$ when n is odd and $\cos^2(\pi/2n)$ when n is even.*

Proof. Consider the cycle graph C_n on n vertices which is a 2-regular graph. Since, the spectra of the unweighted cycle graph is given by $\left\{4\sin^2\left(\frac{\pi r}{n}\right)\right\}_{r=0,\dots,n-1}$ [21], clearly, the spectral radius for C_n is 4 when n is even and $4\cos^2(\pi/2n)$ when n is odd. Similarly, the minimum nonzero eigenvalue of the unweighted Laplacian, $\mathcal{L}$ for C_n is given by $4\sin^2\left(\frac{\pi}{n}\right)$. Suppose the weight on each edge is $\mu > 0$. A sufficient condition for consensus of the agents is that the non-zero eigenvalues of the Laplacian, $\mathcal{L}(C_n)$, lie between $\underline{k}$ and $\bar{k}$. Now, the smallest non-zero eigenvalue of the weighted Laplacian is $4\mu\sin^2\left(\frac{\pi}{n}\right)$ and its spectral radius is 4μ when n is even and $4\mu\cos^2(\pi/2n)$ when n is odd. Hence, for the existence of a feasible edge weight μ, the requirement is that $\frac{\underline{k}}{4\sin^2(\pi/n)} < \mu < \frac{\bar{k}}{4\cos^2(\pi/2n)}$ for odd n and $\frac{\underline{k}}{4\sin^2(\pi/n)} < \mu < \frac{\bar{k}}{4}$ for even n. Ensuring that the upper bound on μ is greater than the lower bound completes the proof. $\square$

B. Robust Consensus: Edge Weight Perturbation

As a dual to the design problem, consider a multi-agent system in (1) and control law (2) with suitably chosen variables $\{\beta_l\}_{l=0,\ldots,m-1}$ and possibly heterogeneous edge weights in the network to satisfy Lemma 1. The robustness question may now be stated.

Problem 2. *For the multi-agent system in* (1) *and* (2)*, subject to Assumption 1, and satisfying the condition in Lemma 1, obtain the range of allowable perturbation, Δ, in edge weight, μ of C_n.*

Some properties of the Laplacian matrix are now stated because these aid in analyzing the effect of edge weight perturbation on the Laplacian spectrum. From Proposition II.1 in [1] for the connected graph $\mathcal{G}$, it follows that the weighted Laplacian matrix, $\mathcal{L}$, is similar to

$$\begin{bmatrix} E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau) RWR^T & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix},$$

where $E(\mathcal{G}_\tau)$ is the incidence matrix for a spanning tree $\mathcal{G}_\tau \subseteq \mathcal{G}$ and the matrix R is given by

$$E(\mathcal{G}) = [E(\mathcal{G}_\tau)\ E(\mathcal{G}_c)] = E(\mathcal{G}_\tau)[I_{n-1}\ T_\tau] = E(\mathcal{G}_\tau)R, \tag{12}$$

where $T_\tau \in \mathbb{R}^{(n-1)\times(|\mathcal{E}|-n+1)}$ may be given by

$$T_\tau = (E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{-1} E(\mathcal{G}_\tau)^T E(\mathcal{G}_c), \tag{13}$$

as in [3] while $\mathcal{G}_c \subset \mathcal{G}$ contains those edges of $\mathcal{G}$ that are not part of $\mathcal{G}_\tau$. The non-zero (positive) eigenvalues of $\mathcal{L}$ are identical to the eigenvalues of $E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau) RWR^T$.

Lemma 3. *The matrix $E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau) RWR^T$ is similar to*

$$(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{\frac{1}{2}} RWR^T (E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{\frac{1}{2}},$$

where $(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{\frac{1}{2}}$ is the positive definite square root of the positive definite matrix $(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))$.

Proof. The matrix $(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))$ is symmetric and positive definite, so it has a symmetric positive definite square root. Let this square root be $M = M^T > 0$ such that $M^2 = E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau)$. Using the similarity transformation $M^{-1}(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau)) RWR^T M$, the result follows. □

The gain margin of the system in (1) or (9) is given by $[\underline{k}, \bar{k}]$ and from Lemma 1 it follows that the eigenvalues of the graph Laplacian are equivalent to the controller gain for the system in Fig. 3. A question similar to Problem 2 was answered for single integrator agents in [1]. However, in [1], for consensus, it sufficed to ensure that the Laplacian, $\mathcal{L}$, remained positive semidefinite when the edge weights were perturbed. But, in the present setup, the following two matrix inequalities need to be satisfied for robust consensus:

$$QWQ^T - \underline{k}\boldsymbol{I}_{n-1} > 0 \tag{14}$$

$$E(\mathcal{G})WE(\mathcal{G})^T - \bar{k}\boldsymbol{I}_n < 0, \tag{15}$$

where $Q = MR = (E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{\frac{1}{2}} R$. This is because (14)-(15) essentially bound the Laplacian spectra above and below so that they belong to $(\underline{k}, \bar{k})$. If the conditions (14)-(15) are satisfied by the system with nominal weights, it is desired to obtain the limit on the perturbation of the edge weights so that any one of the inequalities is violated. A positive perturbation in an edge weight may lead to violation of (15), by increasing the norm of the Laplacian, while a negative perturbation may result in violation of (14). The expressions (14)-(15) are in general true of any undirected graph and not necessarily a cycle. However, for C_n, the bound on edge weight perturbation can be related to the edge weight and other network parameters. Suppose the edge weights on all the edges of the cycle are identical nominally and equal to μ. The main result of this subsection may now be stated.

Theorem 3. *The multi-agent system in* (1)-(2)*, satisfying* (14)-(15)*, is robustly stable against any perturbation, Δ, on the edge weights, μ, that satisfies*

$$\frac{\underline{k}}{4\sin^2(\pi/n)} - \mu < \Delta < \frac{\bar{k}}{4\phi} - \mu \tag{16}$$

where $\phi = 1$ for even n and $\cos^2(\pi/2n)$ for odd n.

Proof. The expressions (14)-(15), taken together, essentially bound the nonzero eigenvalues of the weighted Laplacian of the cycle graph. The maximum and minimum eigenvalues of the Laplacian are given by 4μ (for even n) or $4\mu\cos^2(\pi/2n)$ (for odd n), and $4\mu\sin^2(\pi/n)$, respectively. The perturbed edge weight is given by $\mu + \Delta$ and thus, upon restricting the eigenvalues of the perturbed system to lie in $[\underline{k}, \bar{k}]$, the bound on perturbation is obtained as in (16). □

Remark 5. *The lower bound on the perturbation is negative and the upper bound is positive because the nominal system with edge weight μ satisfies* (14)-(15)*.*

IV. Simulations

Consider a system of inherently unstable identical agents whose linear uncertain dynamics can be described by the characteristic polynomial $s^5 + [16, 20]s^4 + [90, 100]s^3 + [300, 310]s^2 + [340, 350]s + [-40, -35]$. Clearly, Assumption 1 does not hold. Also, let $\beta_i = 1\ \forall i$. From an analysis of stability, using Kharitonov's Theorem, $\underline{k} = 40$ and $\bar{k} = 192.67$. CLearly, using the bound (11) of Theorem 2, it turns out that the maximum number of agents for which the uncertain family of agents achieves consensus is $n = 6$. The range of allowable edge weights is $(40, 48.16)$. Choosing C_6, with $\mu = 44$, the system achieves consensus for an agent model in the interval family, as shown in Fig. 4. Note that the states are not bounded at consensus because Assumption 1 does not hold in this example. With an increase in the number of agents to 7, and a nominal plant in the interval family given by a characteristic polynomial $s^5+19s^4+99s^3+305s^2+345s+-40$, one of the zeros of the polynomial $\bar{P}(s)$ in (8) lies in the right half plane (at 0.0179) and hence, violates the necessary condition for consensus [6]. Also, the range of perturbations, Δ, to the edge weight, μ, can be obtained from Theorem 3 as $-4 < \Delta < 4.16$. Perturbing the edge weight by $+12$ also results in the consensus breaking down, as shown in Fig. 5.

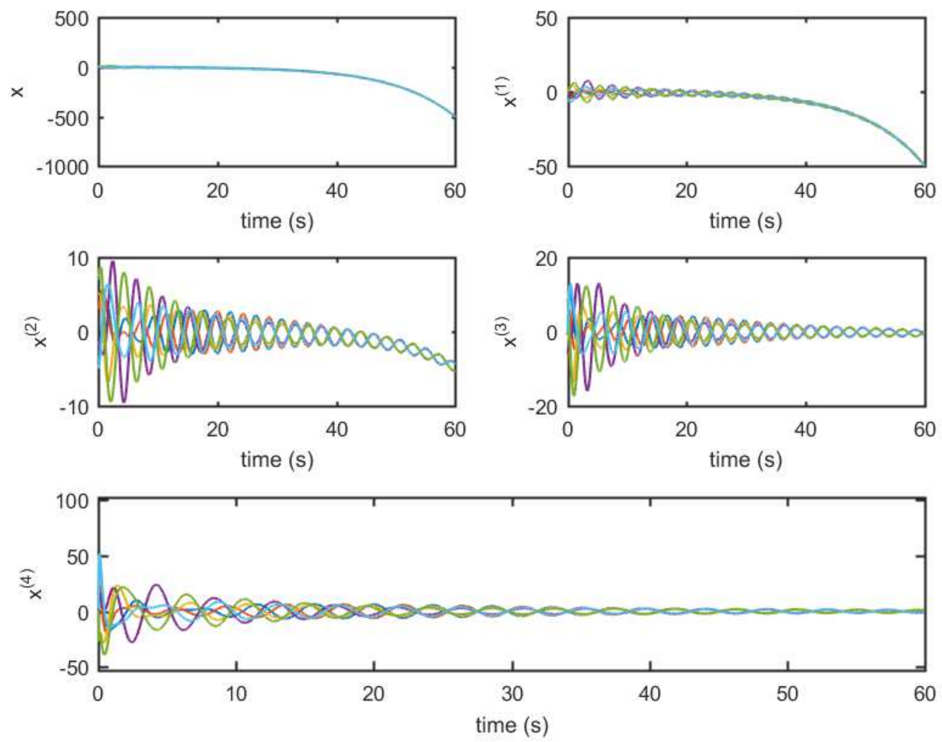

Figure 4: Consensus in states of 5^{th} order uncertain agents over C_6.

V. Conclusions

Using Kharitonov's Theorem as a tool for both design and analysis, this paper provided a relationship between the uncertainty bounds characterized by gain margins and the parameters of the cycle graph C_n. Though the bounds obtained are conservative, as pointed out in Corollary 1, and thus, only provide sufficient conditions for convergence, they give an indication about the dependence of the the tolerable uncertainty margins on the number of agents. The amount of tolerable edge weight perturbation in edge weights of C_n was also derived for the family of uncertain plants. Unlike consensus among integrators, where only negative edge weight perturbations caused divergence among the agents, it was shown that even a positive edge weight perturbation may lead to breakdown of consensus in this case. In future, more general graphs could be considered for similar studies and also uncertain nonlinear systems would provide a more realistic scenario.

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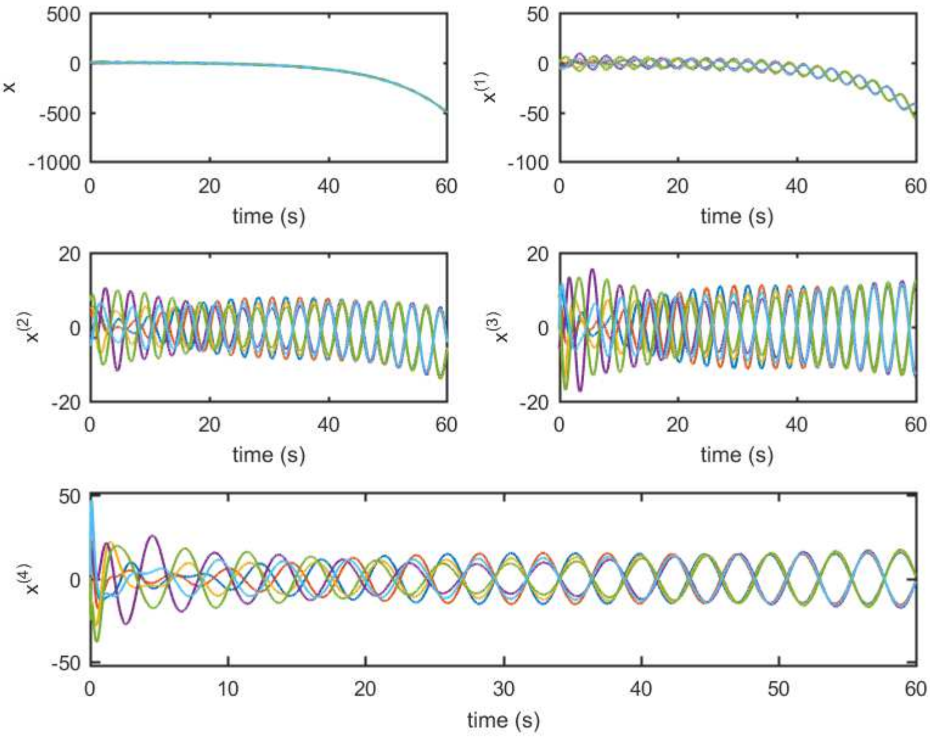

Figure 5: Consensus breaks down with a perturbation in edge weight that violates Theorem 3.

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