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Robustness of Heterogeneous Cyclic Pursuit

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The present paper investigates the robustness of the directed cycle graph using the Nyquist criterion and derives the limit to which a single gain can be varied, while achieving positional consensus among the agents in heterogeneous cyclic pursuit framework. This work builds on similar robustness results for multi-agent systems connected by undirected graphs, in the literature. Simulations back up the theoretical results.

I. Introduction

Cyclic pursuit has been an interesting set up for the study of multi-agent system behaviour for quite some time now [1–8]. Cyclic pursuit is an algorithm where every agent, indexed i, pursues its unique leader, indexed $i+1$ (modulo n) where the total number of agents is n, as shown in Fig. 1. In graph theoretic terms, the agents are running a variation of the consensus protocol over a directed cycle graph [9].

A lot of work has been done on formation control of agents in cyclic pursuit [10, 11]. These have focussed on performing formation maneuvers about certain designated goal points. If this goal point may be extended to a target trajectory, then target capture is also possible using cyclic pursuit. The inherent advantages of cyclic pursuit, stemming from minimal communication requirements for connectivity, can then be gainfully utilized to track a moving target and possibly neutralize the threat. On-line path generator design methodologies combined with cyclic pursuit to enclose a target via geometric formations around the target have been also reported [12]. Some researchers have used vision based cyclic pursuit strategies to capture a moving target in the same sense [13, 14]. A combination of Unmanned Air Vehicles (UAVs) and Unmanned Ground Vehicles (UGVs) to detect targets, using cyclic pursuit, is also to be found in the literature [15]. This certainly requires heterogeneity among agents as mentioned in [16]. In [17], a target capturing strategy is outlined based on local information about the target and one leader agent. In [18], a cyclic pursuit based strategy is proposed to capture a target in minimum time using multiple Micro Air Vehicles (MAVs). It thus turns out that cyclic pursuit may prove to be a very effective strategy in capturing or neutralising threats. Hence, a detailed analysis of cyclic pursuit is necessary to fully harness its potential in such military applications as mentioned above or even in civilian applications as boundary tracking [19].

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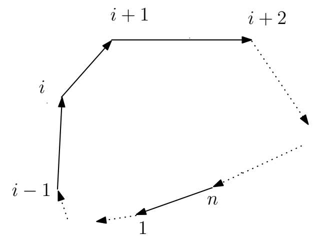


Figure 1: The cyclic pursuit scheme

It has been pointed out in [6] that if one of the gains in cyclic pursuit is negative, the set of points where the agents may achieve rendezvous expands significantly. Thus, from the perspective of the rendezvous problem, it is of interest to study the effect of negative gains in multi-agent systems. This paper builds on the same idea and suggests a broader framework to study general multi-agent systems. The present analysis aims to understand the effect of a single negative gain in the heterogeneous cyclic pursuit framework from a robust stability perspective and goes on to derive the bound on this negative gain using tools of robust control theory.

In this paper, cyclic pursuit will be considered from a graph theoretic perspective, while admitting heterogeneity in the gains of the agents, as in [6]. Some work on the robustness of undirected graphs has been carried out by merging concepts from graph theory and robust control [20, 21], such as application of the small gain theorem to the graph Laplacian and the edge Laplacian matrices. In particular, [21] considered the possibility of admitting negative gains on some of the edges. However, all these works have considered undirected graphs whose Laplacians are symmetric and therefore lend themselves to analysis owing to the special properties of symmetric matrices. In this paper a directed cycle graph, which is the architecture of cyclic pursuit, is considered for the first time for similar robustness studies. While the methods outlined in [21] are suited to the analysis of undirected graphs, it will be shown in this paper that even in the absence of such symmetric Laplacians, the tools outlined therein can be extended to the directed cycle graph. In this paper, the cyclic pursuit system is first transformed to edge variables which denote the difference between the agent states and the uncertainty is introduced in the form of perturbation to one of the edge weights. This can easily be extended to include perturbations to multiple edge weights (agent gains). Once the perturbed system is modelled, with the perturbation in the feedback path, Nyquist criterion is used to evaluate the gain margin. It is shown that this gain margin is the amount of perturbation that any individual gain can withstand and this limit on the perturbation agrees with the result obtained in [6]. However, the present framework can also handle frequency dependent perturbations and is thus a more generic framework. Thus, the analysis presented here holds out promise for analyzing more general structures of directed graphs.

Section II provides some preliminary results on directed graphs while Section III deals presents some results on the directed cycle graph that represents the cyclic pursuit system. Section IV presents the results on the robust stability of the weighted cycle graph by considering one of the edge weights to be uncertain. In Section V, some simulation results are provided that corroborate the results of Section IV. Finally, Section VI concludes the paper.

Notation The null space of a matrix A is denoted by $\mathcal{N}(A)$. The vector of all-ones is denote by $\mathbf{1}$ with or without a subscript. The subscript denotes the dimension of the space to which it belongs and wherever the subscript is missing, the dimension is to be understood from the context. Similarly, $\mathbf{0}$ denotes a vector of all-zeros with the dimension understood from the context. A graph $\mathcal{G}$ is specified by its vertex set V and the edge set $\mathcal{E}$ whose members capture the incidence relation between pairs of V. In this paper the cardinalities of V and $\mathcal{E}$ are chosen to be n and m, respectively.

II. Preliminaries on Directed Graphs

This works utilizes many notions from algebraic graph theory [22]. Of primary interest for this work are the graph Laplacian matrix, L_g, defined over a directed graph, denoted $\mathcal{G} = (V, \mathcal{E})$, and the edge Laplacian matrix, denoted L_e, introduced in [20,21]. One of the main features of the graph and edge Laplacian matrices for the case of undirected graph is that they are symmetric and positive semidefinite. However, for directed graphs, these properties do not hold.

Using a similar approach defined in [20,21], the graph Laplacian for a directed graph is defined as $L_g = \mathcal{A}E^T$, where $\mathcal{A} \in \mathbb{R}^{n\times m}$ is such that $\mathcal{A}_{ij} = 1$ if the edge j has its tail at vertex i and is 0 otherwise, and the incidence matrix E is defined such that $[E]_{ij} = 1$ if edge j begins at vertex i, $[E]_{ij} = -1$ if edge j terminates at vertex i, and $[E]_{ij} = 0$ otherwise. It is at once apparent that the graph Laplacian, so defined for a digraph, is not symmetric.

It is customary to represent the agents (UAVs or UGVs) in a multi-agent system by vertices of a directed graph and the connections between them as edges. It will be clear later that the kinematics along the x and y- directions are identical for the cyclic pursuit framework. So it is sufficient to consider motion along any one of them to conclude similar results about the other. In this paper, the evolution of the motion along the x-direction is considered and thus the states of the n agents can be represented by the vector $x_v \in \mathbb{R}^n$ to analyze cyclic pursuit in a plane. The edge states are the differences between the vertex states that are connected by directed edges, given by $x_e = E^T x_v \in \mathbb{R}^m$. Since most consensus protocols use the dynamics $\dot{x}_v(t) = -L_g x_v(t)$, in terms of the edge dynamics, this may be represented as $\dot{x}_e(t) = -E^T\mathcal{A}x_e(t)$. Now, defining this system matrix $-E^T\mathcal{A} = -L_e$, the negative of the edge Laplacian, one may arrive at the following relation:

$$E^T L_g = L_e E^T. \tag{1}$$

The following result is straightforward and stated without proof.

Lemma 1. *The null space of $\mathcal{A}$, ($\mathcal{N}(\mathcal{A})$) is a subset of $\mathcal{N}(L_e)$, that is $\mathcal{N}(\mathcal{A}) \subseteq \mathcal{N}(L_e)$.*

Lemma 2. *The following statements are equivalent:*

1. *$\mathcal{A}$ has a nontrivial null space.*
2. *$\mathcal{A}$ has at least two identical columns.*
3. *The out-degree of at least one vertex in $\mathcal{G}$ is greater than unity.*

Proof. If $\mathcal{A}$ has a nontrivial null space, it means that its columns are not linearly independent. Now, each column of $\mathcal{G}$ corresponds to an edge of $\mathcal{G}$. Suppose each edge begins from a different vertex. Then there can be no more than n edges and the columns of $\mathcal{G}$ will be a permutation of the columns of the identity matrix I_n and hence of full rank (trivial null space). But, if there are multiple edges emanating from any vertex (out-degree of the corresponding vertex being greater than unity), the columns of $\mathcal{G}$ corresponding to these edges will be identical. Hence, a nontrivial null space results. Thus, $3 \Leftrightarrow 1 \Leftrightarrow 2$. □

Lemma 3. *If there are r vertices in $\mathcal{G}$ each of whose out-degree is greater than or equal to one, then $dim[\mathcal{N}(L_e)] \geq m - r$.*

Proof. Suppose there are r vertices from which there is at least one outgoing edge. Then, each row of $\mathcal{A}$ corresponding to these r vertices, has at least one nonzero element (1, to be precise), at unique locations. This is because the position of the non zero elements correspond to an outgoing edge and an edge can only emanate from a unique vertex. Thus, all these r rows are linearly independent. Hence, $rank(\mathcal{A}) = r$. By rank-nullity theorem, $dim[\mathcal{N}(\mathcal{A})] = m - r$. Now, by Lemma 1, $\mathcal{N}(\mathcal{A}) \subseteq \mathcal{N}(L_e)$. This implies that $m - r = dim[\mathcal{N}(\mathcal{A})] \leq dim[\mathcal{N}(L_e)]$. Hence the proof. □

Lemma 4. *For a weakly connected directed graph, $\mathcal{N}(L_e) = \mathcal{N}(\mathcal{A})$ unless the all-ones vector, $\mathbf{1_n}$, belongs to the range space of $\mathcal{A}$, which is only feasible if the out degree of every node is greater than or equal to one.*

Proof. Suppose the all-ones vector $\mathbf{1_n}$ belongs to the range space of $\mathcal{A}$. This means that there exists a vector $x \in \mathbb{R}^m$ such that $\mathcal{A}x = \mathbf{1_n}$. Now, it is known that $E^T\mathbf{1_n} = \mathbf{0}$ since every edge begins and ends at exactly one node and thus the column sum for every column of E is zero. Hence, $L_e x = E^T\mathcal{A}x = E^T\mathbf{1_n} = \mathbf{0}$ even though x clearly does not belong to the null space of $\mathcal{A}$.

Suppose the all-ones vector does not belong to the range space of $\mathcal{A}$. Now, if $L_e x = \mathbf{0}$ for some non trivial x, then either $\mathcal{A}x = \mathbf{0}$ or $\mathcal{A}x$ belongs to the null space of E^T. But, for a weakly connected graph, the rank of E^T is $n-1$ and hence the null space of E^T is of dimension 1, spanned by the all-ones vector. Thus, it is required that $\mathcal{A}x = \mathbf{1_n}$ for some $x \in \mathbb{R}^{>}$. But, $\mathcal{A}x$ can never equal the all-ones vector for any x. Hence, the null space of L_e contains only those vectors that are also in the null space of $\mathcal{A}$ and vice versa. □

From the preceding lemmas, it may be recognised that either a vertex with an out-degree greater than unity or the presence of appropriate cycles (directed or undirected) in a digraph $\mathcal{G}$ result in a nontrivial null space of L_e.

III. Cyclic Pursuit: A Graph Theoretic Perspective

In the conventional cyclic pursuit, the velocity of agent i is proportional to the relative position of its leader, agent $i+1$, to itself. The constant of proportionality is called the gain (k_i) and is conventionally chosen to be the same for every agent. Heterogeneous linear cyclic pursuit can be mathematically expressed through the kinematics given by:

$$\dot{p_i} \quad = \quad k_i(p_{i+1} - p_i), \;\; k_i > 0 \; \forall i \tag{2}$$

where the position of agent i is given by $p_i \in \mathbb{R}^2$ and the indices i, $i+1$ are chosen modulo n, the number of agents. This protocol has been shown to result in positional consensus among the agents, that is rendezvous, when all the gains (k_i) are positive [6]. Equation

(2) justifies considering the motion along the x- direction only throughout this paper, as the equation of motion along the y-direction is clearly similar.

The analysis in this section is along the lines followed in [20]. The basic idea is to consider the edge variables (the differences between the states of the vertices/agents) which must reduce to zero, for consensus. It will be shown that it suffices to consider the spanning tree for analysis of the consensus of the cycle graph since the other states are nothing but linear combinations of the edge states of the spanning tree. Thus, the overall cyclic pursuit system in (2) will be expressed in terms of the edge variables and the spanning tree will be considered for agreement. The cycle graph clearly contains the same number of edges as the number of vertices. Thus $m = n$. The incidence matrix $E(\mathcal{G}) \in \mathbb{R}^{n \times n}$ for the directed cycle graph with n vertices and edges is given by:

$$E(\mathcal{G}) = \begin{bmatrix} 1 & 0 & 0 & \dots & -1 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{3}$$

It is easy to see that the spanning tree for cyclic pursuit can be obtained by deleting any one edge from the directed cycle graph. This spanning tree graph obtained from $\mathcal{G}$ by this deletion is denoted by $\mathcal{G}_\tau$. For a directed cycle graph with n vertices (and edges), let the last edge, joining vertex n with vertex 1 (along the same direction), denoted by e_n, be deleted to obtain the spanning tree $\mathcal{G}_\tau$. The incidence matrix of this spanning tree, $E(\mathcal{G}_\tau) \in \mathbb{R}^{n \times n-1}$ is given by:

$$E(\mathcal{G}_\tau) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & -1 \end{bmatrix}. \tag{4}$$

Thus, from (3) and (4), the incidence matrix of the graph may be related to that of the spanning tree by the following relation:

$$E(\mathcal{G}) = E(\mathcal{G}_\tau)[I_{n-1} \quad T_\tau^c] = E(\mathcal{G}_\tau)R(\mathcal{G}), \tag{5}$$

where, $T_\tau^c = -\mathbf{1_{n-1}} \in \mathbb{R}^{n-1}$. Thus, $R(\mathcal{G}) \in \mathbb{R}^{(n-1)\times n}$ is given by:

$$R(\mathcal{G}) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 \\ 0 & 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix}. \tag{6}$$

Hence, $R(\mathcal{G})\mathbf{1}_n = \mathbf{0}_n$. By the same token, $\mathcal{A}(\mathcal{G}_\tau) = [I_{n-1} \quad \mathbf{0}]^T$, where, $\mathbf{0} \in \mathbb{R}^{n-1}$. It maybe readily verified that for the spanning tree, the edge Laplacian, $L_e(\mathcal{G}_\tau) =$

$E(\mathcal{G}_\tau)^T\mathcal{A}(\mathcal{G}_\tau) \in \mathbb{R}^{(n-1)\times(n-1)}$ is given by:

$$L_e(\mathcal{G}_\tau) = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{7}$$

Consider the graph Laplacian and the edge Laplacian for the weighted cycle digraph given by $L_g = \mathcal{A}WE(\mathcal{G})^T$ and $L_e = E(\mathcal{G})^T\mathcal{A}W$, respectively. Here, $W \in \mathbb{R}^{n\times n}$ is a diagonal matrix, whose diagonal entries are the weights of the corresponding edges, that is $W_{ii} = k_i$, where k_i is the gain of agent i in cyclic pursuit (as described by (2)). As stated before, it is sufficient to consider motion along the x-direction only. Once again, (5) holds as before with $T_\tau^c = -\mathbf{1} \in \mathbb{R}^{n-1}$, while $\mathcal{A} = I_n$. It may be readily verified from (4) and (6), respectively that $E(\mathcal{G}_\tau)^TWE(\mathcal{G}_\tau)$ and $R(\mathcal{G}_\tau)WR(\mathcal{G}_\tau)^T$ are invertible matrices when all the edge weights (agent gains) are positive real numbers. Results similar to Theorems 2.5-2.7 in [20] will now be stated and proved for the weighted cycle graph with n edges and n nodes.

Lemma 5. *The graph Laplacian for weighted cyclic pursuit, $L_g = \mathcal{A}WE(\mathcal{G})^T$ is similar to* $\begin{bmatrix} E(\mathcal{G}_\tau)^TWR(\mathcal{G})^T & 0 \\ 0 & 0 \end{bmatrix}$.

Proof. Consider the matrices $S_1 = [WE(\mathcal{G}_\tau)(E(\mathcal{G}_\tau)^TWE(\mathcal{G}_\tau))^{-1} \quad \mathbf{1}_n\left(1/\sum_i \frac{1}{k_i}\right)]$ and $S_1^{-1} = \begin{bmatrix} E(\mathcal{G}_\tau)^T \\ \mathbf{1}_n^TW^{-1} \end{bmatrix}$. It is at once apparent that $S_1^{-1}L_gS_1 = \begin{bmatrix} E(\mathcal{G}_\tau)^TWR(\mathcal{G})^T & 0 \\ 0 & 0 \end{bmatrix}$. Hence the proof. □

Lemma 6. *The edge Laplacian for weighted cyclic pursuit, $L_e = E(\mathcal{G})^T\mathcal{A}W$ is similar to* $\begin{bmatrix} E(\mathcal{G}_\tau)^TWR(\mathcal{G})^T & 0 \\ 0 & 0 \end{bmatrix}$.

Proof. Consider the matrices $S_2 = [R(\mathcal{G})^T \quad W^{-1}\mathbf{1}_n]$ and $S_2^{-1} = \begin{bmatrix} (R(\mathcal{G}_\tau)WR(\mathcal{G}_\tau)^T)^{-1}R(\mathcal{G}_\tau)W \\ \left(1/\sum_i \frac{1}{k_i}\right)\mathbf{1}_n^T \end{bmatrix}$

It may be verified that $S_2^{-1}L_eS_2 = \begin{bmatrix} E(\mathcal{G}_\tau)^TWR(\mathcal{G})^T & 0 \\ 0 & 0 \end{bmatrix}$. Hence the proof. □

Lemma 7. *For the weighted cycle digraph, the edge Laplacian is similar to the graph Laplacian.*

Proof. Since both the graph Laplacian, L_g and the edge Laplacian, L_e for the cyclic digraph are similar to the same matrix given by $\begin{bmatrix} E(\mathcal{G}_\tau)^TWR(\mathcal{G})^T & 0 \\ 0 & 0 \end{bmatrix}$, hence using the transformation $S^{-1}L_eS$, with $S = S_2S_1^{-1}$, the result follows immediately. □

Suppose the edge states of the spanning tree are represented by $x_\tau(t)$ and the remaining edge is given by $x_a(t)$. The edge version of cyclic pursuit, in view of the Lemmas 5-7, can be written as

$$\begin{bmatrix} \dot{x}_\tau(t) \\ \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} -E(\mathcal{G}_\tau)^T W R(\mathcal{G})^T & 0, \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_\tau(t) \\ x_a(t) \end{bmatrix}. \tag{8}$$

IV. Heterogeneous Cyclic Pursuit: A Weighted Laplacian for a Digraph

In this section, the robust stability of a cyclic pursuit system will be investigated with respect to positional consensus. It has already been shown in (8) that it is sufficient to consider the dynamics of the spanning tree to investigate positional consensus. The uncertainty in the cyclic pursuit system may be due to perturbations on the gains of the agents, which, from a graph theoretic perspective, are essentially the weights on the edges. For perturbations on multiple edges, small gain theorem may be required. However, in this paper only one of the gains is assumed to be uncertain.

A. Cyclic pursuit with uncertainty

The linear cyclic pursuit protocol may be considered in the presence of exogenous finite energy disturbances. In the light of Lemmas 5 through 7, and (8), it may be observed that for the edge agreement protocol, it is sufficient to concentrate on the dynamics of the reduced spanning tree given by

$$\dot{x}_\tau(t) = -E(\mathcal{G}_\tau)^T W R(\mathcal{G})^T x_\tau(t) + E(\mathcal{G}_\tau)^T v(t), \tag{9}$$

$$z(t) = E(\mathcal{G}_0)^T W E(\mathcal{G}_\tau)(E(\mathcal{G}_\tau)^T W E(\mathcal{G}_\tau))^{-1} x_\tau(t), \tag{10}$$

where, $\mathcal{G}_0$ is graph representing the observed edge variables or states, $z(t)$ are the observed edge states, $x_\tau(t)$ are the edge states of the spanning tree and $v(t) \in \mathbb{R}^n$ is the vector that represents a finite energy exogenous disturbance entering each agent (vertex). This set up is suited for analysis using small gain theorem using upper fractional transformation. However, in this paper the approach will focus on representing the system as a Single Input- Single Output feedback configuration. So, the exogenous input $v(t)$ and the observed edge states $z(t)$ will not be considered here.

The notion of uncertainty is now introduced through the edge weights. Suppose the weights on one of the n edges is uncertain. The perturbations are bounded about some nominal value of the edge weight or gain and this nominal value is positive. For tractability of the problem, the perturbation is assumed in any one of the edge weights (say in k_1, without loss of generality). In case of multiple uncertain weights, small gain theorem may be used. The general framework may be described as follows. The uncertainty on any edge weight k_i is an additive one, δ_i, incorporated as $k_i + \delta_i$ where, $|\delta_i| < \bar{\delta}, \forall i$. The uncertainty set is then defined as

$$\boldsymbol{\Delta} = \{\Delta : \Delta = \delta_{\mathbf{1}}, ||\delta_{\mathbf{1}}||_\infty \leq \bar{\delta}\}. \tag{11}$$

Now, the uncertain edge agreement protocol may be described as

$$\dot{x}_\tau(t) = -E(\mathcal{G}_\tau)^T (W + P\Delta P^T) R(\mathcal{G})^T x_\tau(t), \tag{12}$$

with the uncertainties belonging to the set given by (11) and $P \in \mathbb{R}^{n\times 1}$ is a $\{0, 1\}$ vector with 0-entries everywhere except at $P_{1,1}$.

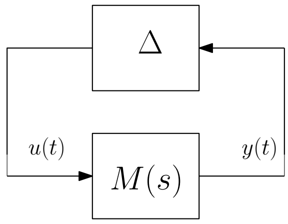


Figure 2: Cyclic pursuit with uncertain gain

B. Robust stability of uncertain cyclic pursuit protocol

In the literature [6] it has been proved using Gershgorin's theorem that for cyclic pursuit with heterogeneous but positive gains, the agents will rendezvous to a point. Moreover, even with at most one negative gain, subject to a lower limit, the system will still arrive at positional consensus. The limit has also been derived in [6]. In this section, an alternate robust stability approach, using the gain margin, has been employed to derive the same limit on the negative gain as in [6]. Moreover, in this paper the problem has been cast within a more general framework which may admit multiple uncertainties. The uncertain system, described by (12), is to be transformed in such a way that the uncertainty is separated from the nominal plant as illustrated in Fig. 2.

This formulation lends itself easily to a stability analysis using the Nyquist criterion and the idea of gain margin. The next step, therefore, involves the determination of $M(s)$. It is assumed, without loss of generality, that the only uncertain gain is k_1 and the exogenous signal $v(t)$ is zero. Next, consider the input $u(t)$ and output $y(t)$ of the plant to be given by

$$u(s) = \Delta y(s), \tag{13}$$
$$y(s) = M(s)u(s), \tag{14}$$

and the overall system described by

$$\dot{x}_\tau(t) = -E(\mathcal{G}_\tau)^T W R(\mathcal{G})^T x_\tau(t) - E(\mathcal{G}_\tau)^T P u(t), \tag{15}$$
$$y(t) = P^T R(\mathcal{G})^T x_\tau(t), \tag{16}$$
$$u(t) = \Delta P^T R(\mathcal{G})^T x_\tau(t). \tag{17}$$

This is the model depicted in Fig. 2 and the transfer function, $M(s)$, between $y(s)$ and $u(s)$ can be written as:

$$M(s) = -P^T R(\mathcal{G})^T (sI + E(\mathcal{G}_\tau)^T W R(\mathcal{G})^T)^{-1} E(\mathcal{G}_\tau)^T P. \tag{18}$$

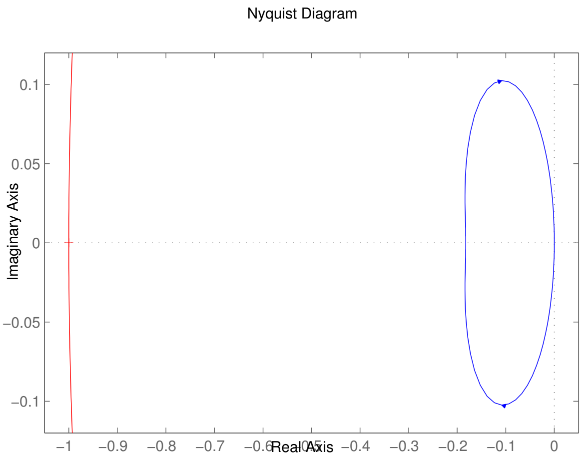
When (18) is combined with (13)-(14), the loop is closed in Fig. 2. Now, the uncertainty Δ is a scalar in this case and since $M(s)$ is a Single Input-Single Output system, a classical Nyquist analysis of the gain margin will lead to results on the stability. After some algebraic manipulations, it transpires that $-M(s)$ can be obtained as the $(1, 1)$ entry of the matrix D_{n-1}^{-1} where, D_{n-1} is given by:

$$D_{n-1} = \begin{bmatrix} (s+k_1) & -k_2 & 0 & \dots & 0 \\ 0 & (s+k_2) & -k_3 & \dots & 0 \\ 0 & 0 & (s+k_3) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_n & k_n & k_n & \dots & (s+k_{n-1}+k_n) \end{bmatrix}. \tag{19}$$

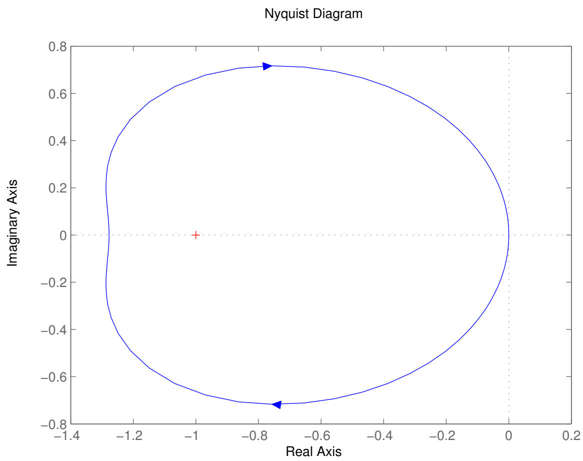
Define the matrix obtained by removing the j-th column and i-th row of any matrix A as $A_{(i,j)}$. A recursive relation on D_{n-1} results, as follows:

$$\det D_{n-1} = (s+k_1)\det D_{n-1(1,1)} + k_2 \det D_{n-1(1,2)}. \tag{20}$$

It can be checked that $\det D_{n-1(1,2)} = \Pi_{j=3}^{n} k_j$. Thus, the $\{1,1\}$ entry of D_{n-1}^{-1} is given by $-M(s) = \frac{\det D_{n-1(1,1)}}{(s+k_1)\det D_{n-1(1,1)} + k_2 \det D_{n-1(1,2)}}$. It may be observed that $M(s)$ has a relative degree one and so the Nyquist plot of $M(s)$ starts at the negative real axis when $\omega = 0$ and approaches the origin along the positive imaginary axis as $\omega \to \infty$. A typical Nyquist plot for a system of five agents (four of the fixed gains are [4 3 2 1] and the fifth gain's nominal value is 5) is shown in Fig. 3a. Since Δ is a negative real number, (for positive perturbations the system will always achieve consensus and therefore remain stable), so it is effectively a negative feedback. Therefore, in order to ensure stability, one needs to ensure that the point $(-1,0)$ is not encircled by the Nyquist plot of $M(s)$. The amount of constant perturbation that the system can withstand before losing stability can be obtained straightaway from the gain margin. For example, in Fig. 3a, this margin is $1/-0.1825 = -5.48$. This means that the fifth gain can be perturbed from its nominal value of 5 down to -0.48, before the system loses stability. Since the relative degree of $M(s)$ is one and it is of type zero, the phase crossover frequency is zero. Thus, gain margin can be obtained by evaluating $1/M(0)$. Similarly, when the magnitude of the perturbation is -7, that is, the uncertain gain is brought down to -2 from 5, the critical point, $(-1,0)$, is encircled by the Nyquist plot as shown in Fig. 3b.



(a)



(b)

Figure 3: (a) Typical Nyquist plot of $M(s)$ for a system with five agents and one uncertain gain (stable) within bound and (b) with perturbation violating bound.

It is immediately apparent that $M(0) = -\dfrac{\sum_{i=2}^{n} \frac{1}{k_i}}{1 + k_1 \sum_{i=2}^{n} \frac{1}{k_i}}$ (by substituting $s = 0$ and using the recursive relation (20)). Now, applying the Nyquist criterion, with $\bar{\delta}$

representing the bound on the uncertainty in k_1, the following expression results:

$$-k_1 - \frac{1}{\sum_{i=2}^{n} \frac{1}{k_i}} < \bar{\delta}$$
$$\Rightarrow k_1 + \bar{\delta} > -\frac{1}{\sum_{i=2}^{n} \frac{1}{k_i}}. \tag{21}$$

The following theorem summarizes the main result in this paper.

Theorem 1. *Given a perturbation on a single edge, say edge j (with nominal weight k_j), the heterogeneous cyclic pursuit, described by (2), is stable for all perturbations greater than $\bar{\delta}$ given by:*

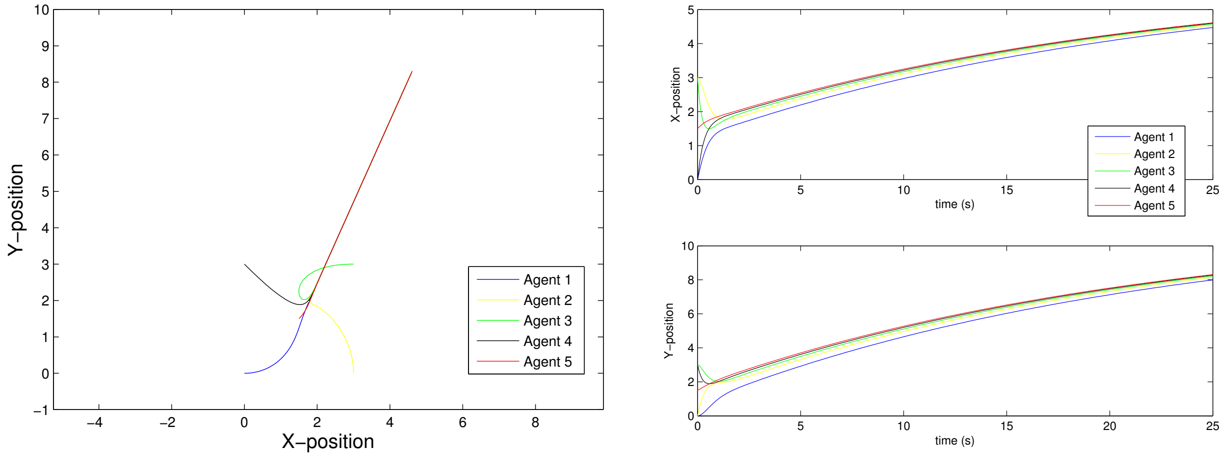
$$\bar{\delta} > -k_j - \frac{1}{\sum_{i=1, i\neq j}^{n} \frac{1}{k_i}}. \tag{22}$$

It should be noted that this is the same bound derived in [6] as the necessary and sufficient condition for the stability of cyclic pursuit and is thus not a conservative bound. For the cycle graph, this is also the equivalent resistance as seen between the vertices j and $j+1$ when the edge joining j and $j+1$ is removed, where, the reciprocal of the edge weight is the resistance corresponding to each edge. It has been shown, using the small gain theorem, in [21] that for the undirected graph, an edge weight can be perturbed to have negative values and so long as this negative value is greater than a lower bound, consensus will still be achieved. In fact, [21] shows that in the undirected case, this bound is exactly equal to the negative of the equivalent resistance (with the corresponding edge removed) between the vertices that the edge joins. Also, it has been shown that when only one edge weight is uncertain, this bound is not a conservative one. The result presented in this paper now shows that even though the graph corresponding to cyclic pursuit is a directed one, the same result holds good here too.

In case there are frequency dependent perturbations or multiple uncertain edge weights, this same framework, developed here, can be used for robustness studies using small gain theorem. Using an upper fractional transformation, the system (9)-(10) can be split up into the M-Δ form, whereafter the small gain theorem may be applied. It turns out that the condition for stability in that case would have been $||M(j\omega)\Delta(j\omega)|| < 1$ and $\Delta(j\omega)$ would be a vector of uncertain weights dependent on frequencies. Since, only the upper bound of $||\Delta(\omega)||$ is likely to be known, in general, the condition is further refined to $||M(j\omega)||||\Delta(j\omega)|| < 1$. Though this makes the problem more tractable, this is bound to lead to conservative results, even for one uncertain edge weight. This is mainly because, $||M(j\omega)\Delta(j\omega)|| \leq ||M(j\omega)||||\Delta(j\omega)||$, with equality resulting only when the peaks of the perturbation and the plant frequency responses coincide at the same frequency and that frequency is not necessarily zero. Hence, the Nyquist approach, used in this paper, yields the tightest bound for the frequency independent perturbation. However, the framework presented in this paper also paves the way for analysis via the small gain theorem. One needs to factorize the plant into using the upper fractional transformation from (9)-(10). In that case the bound will also depend on the frequency response of $M(s)$, unlike in the present paper where it suffices to consider only $M(0)$.

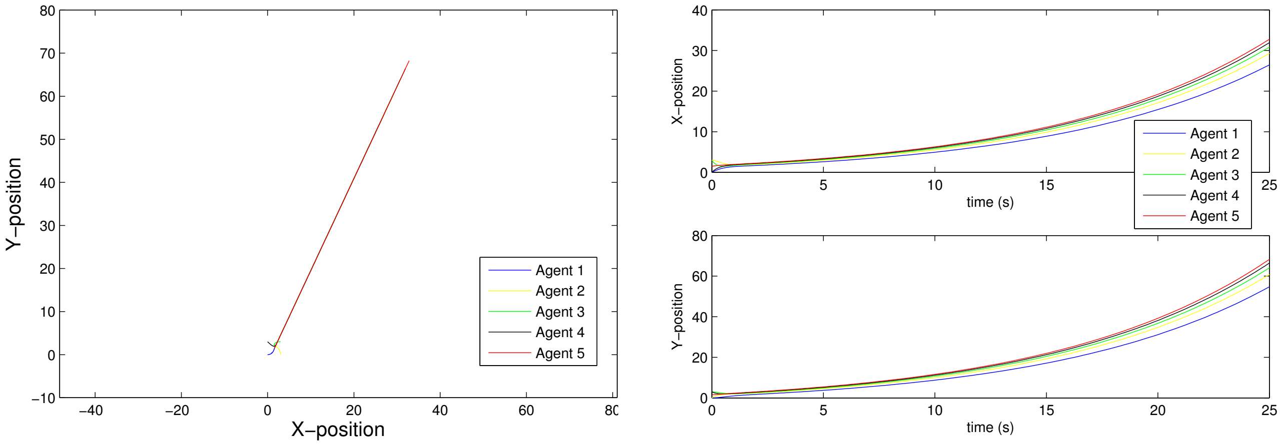
V. Simulation Results

In this section a multi-agent system, comprising five agents, is chosen. The gains of four agents are fixed beforehand, that is $[k_1\ k_2\ k_3\ k_4] = [1\ 2\ 3\ 4]$. The fifth agent is allowed to have a negative gain k_5. In the first example, $k_5 = -0.45$, though negative, satisfies the bound specified in Theorem 1, by (22) (this bound is -0.48, as obtained from the Nyquist plot in Fig. 3a earlier), and leads to positional consensus, as shown in Figs. 4a and 4b. In the second example, the same bound is violated as $k_5 = -0.55$ and it is observed that the agents diverge instead of converging on a rendezvous point, as shown in Figs. 5a and 5b . The initial positions for the agents in both the cases are $(0, 0)$, $(3, 0)$, $(3, 3)$, $(0, 3)$ and $(1.5, 1.5)$ respectively.



(a) (b)

Figure 4: (a) Trajectories for convergent cyclic pursuit and (b) evolution of agent positions with time.



(a) (b)

Figure 5: (a) Trajectories for divergent cyclic pursuit and (b) evolution of agent positions with time.

VI. Conclusions

In conclusion, it may be stated that while robustness tools for undirected graphs have been developed to a great extent, there is a need for developing the same for the directed graphs in order to aid the analysis of multi-agent systems. As a first step, this paper presents some results for the directed cycle graphs. The obtained results are in agreement with the ones in the existing literature. It is hoped that this will serve as encouragement to pursue the robustness studies of directed graphs along the avenues outlined in this paper.

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