Retraction Balancing and Formation Control

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Abstract— We consider a formation control problem in which a collection of systems is ought to attain a balanced configuration on a submanifold of their state space. The submanifold thus determines the shape and position of the desired formation. We solve the formation control problem by simultaneously balancing the retractions of the systems onto the submanifold and asymptotically stabilizing the submanifold. In doing so, we arrive at a distributed control law.

I. INTRODUCTION

We consider formation control problems in which a group of $\mathscr{V} = \{1 \cdots n\}$ systems must reach a balanced (i.e. equidistant) configuration on a target submanifold $M \subset \mathbb{R}^m$ of their state space. Throughout the manuscript, we let M determine the shape and position of the formation. This is similar to the problem studied by Sepulchre, Paley, and Leonard [1], [2] for M being a circle (also cf. [3]–[5]), but we admit for rather general submanifolds.

For solving this problem, we employ balancing algorithms, tubular neighborhoods, and submanifold stabilization.

Balancing algorithms have been studied in detail by Scardovi, Sarlette, and Sepulchre [6], [7] for systems that live on a manifold (for the dual consensus problem on a manifold, we refer the reader either to Sarlette, and Sepulchre [7] for the extrinsic point of view, or to Tron, Afsari, and Vidal [8] for the intrinsic point of view). We, however, admit for the systems to live in the ambient space $\mathbb{R}^m$ of M and thus must balance the retractions of the systems onto the submanifold rather than the actual position of the systems.

The retraction onto a submanifold, however, is only defined within its tubular neighborhoods. This is a direct consequence of the tubular neighborhood theorem, which we employ (see [9, section II.11] or [10, chapter 10]).

At the same time, we bring the systems to M asymptotically by rendering M an asymptotically stable invariant set of the systems, such as in our previous work [11].

As we employ retractions, our solution is restricted to tubular neighborhoods. Such topological obstructions to global solutions are common in formation control and have quite recently been studied by Belabbas [12].

The paper is structured as follows: We formalize our problem in section II, propose a control law whose convergence properties we study in section III, study two examples in sections IV-V, and conclude the paper with section VI.

II. PROBLEM SETUP

We consider a compact, smoothly embedded submanifold $M \subset \mathbb{R}^m$, which here and henceforth defines the shape and position of the formation to be attained. Let $\gamma_a^b : [0,1] \to M$ denote a (minimizing) geodesic on M such that $\gamma(0) = a$ and $\gamma(1) = b$. Then the length functional of γ_a^b, which we denote by $\ell\left(\gamma_a^b\right)$ endows M with the properties of a metric space. If we moreover introduce a weighted, undirected graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{W})$, i.e.

$$\mathscr{V} = \{1 \cdots n\} \subset \mathbb{N},$$
$$\mathscr{E} \subset (\mathscr{V} \times \mathscr{V}) \quad \text{symmetric,}$$
$$\mathscr{W} : \mathscr{E} \to (0, \infty), \quad (i,j) \mapsto w_{ij} \quad \text{symmetric,}$$

then the function

$$P : M^n \to \mathbb{R}, \quad x \mapsto \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{w_{ij}}{4} \ell\left(\gamma_{x_i}^{x_j}\right)^2 \tag{1}$$

(here and henceforth, we replace $(x_1 \cdots x_n)$ by x) defines the so-called diffusive couplings

$$x \mapsto -\operatorname{grad} P(x), \tag{2}$$

where the gradient operator is defined such that it must suffice

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} P(\gamma(s))\right|_{s=0} = \operatorname{grad} P(x) \cdot \left.\frac{\mathrm{d}}{\mathrm{d}s}\gamma(s)\right|_{s=0} \tag{3}$$

for any $\gamma : (-\epsilon, \epsilon) \to M^n$ with the properties $\gamma(0) = x$ and $\left.\frac{\mathrm{d}}{\mathrm{d}s}\gamma(s)\right|_{s=0} \in \mathcal{T}_x M^n$, where $\mathcal{T}_x M^n$ is the tangent space of M^n at x. The convergence properties of the differential equation

$$\dot{x} = -\operatorname{grad} P(x) \tag{4}$$

have thoroughly been studied [8] and, in particular, it has been shown that solutions to (4) locally approach the so-called synchronization manifold

$$\{x \in M^n | x_1 = \cdots = x_n\}. \tag{5}$$

The latter is called the consensus problem. On the other hand, the differential equation

$$\dot{x} = \operatorname{grad} P(x) \tag{6}$$

and the convergence of its solutions to the maximizer of P was studied [6], [7], which is referred to as balancing (dual to consensus). In this latter spirit, define

$$p_0 = \max_{x \in M^n} P(x). \tag{7}$$

Existence of this maximum can be concluded from the extreme value theorem as P is continuous and M is compact. We denote the arguments at which P attains p_0 by

$$B = \{x \in M^n | P(x) = p_0\}. \qquad (8)$$

Elements of B are called *balanced configurations*. Please note that B is parameterized by the choice of $\mathscr{G}$. However, we do not want to go into detail about the particular shape of B in dependence on $\mathscr{G}$ as this has already been studied by Scardovi, Sarlette, and Sepulchre [6], [7]. Accordingly, also define the function

$$\bar{P} : M^n \to \mathbb{R}, \quad x \mapsto p_0 - P(x) \qquad (9)$$

and its sublevel sets

$$U_{\bar{P}}^{\alpha} = \{x \in M^n | \bar{P}(x) \leq \alpha\} \qquad (10)$$

with the identity $U_{\bar{P}}^0 = B$.

We next introduce *tubular neighborhoods*: Let U_M^{ϵ} denote the neighborhood $\{y \in \mathbb{R}^m | \ d(M, y) \leq \epsilon\}$ of M. The normal space of M at x_i will be denoted by $\mathcal{N}_{x_i} M$ and is defined to be the orthogonal complement of $\mathcal{T}_{x_i} M$ in $\mathbb{R}^m$. The normal bundle of M is denoted by $\mathcal{N}M = \{(x_i, y_i) \in M \times \mathbb{R}^m | y_i \in \mathcal{N}_{x_i} M\}$. We define the subset $U_{\mathcal{N}M}^{\epsilon}$ of $\mathcal{N}M$ by $U_{\mathcal{N}M}^{\epsilon} = \{(x_i, y_i) \in \mathcal{N}M | \|y_i\| \leq \epsilon\}$, the projection π from $\mathcal{N}M$ to M by $\pi(x_i, y_i) = x_i$, and the map $\rho : \mathrm{int}\, U_{\mathcal{N}M}^{\epsilon} \to \mathbb{R}^n$, $(x_i, y_i) \mapsto x_i + y_i$ (here and henceforth, int denotes the interior of a set).

Definition 1 (Tubular Neighborhood): The neighborhood $\mathrm{int}\, U_M^{\epsilon}$ of M is said to be a tubular neighborhood of M if it is the diffeomorphic image of $\rho : \mathrm{int}\, U_{\mathcal{N}M}^{\epsilon} \to \mathbb{R}^m$.

This lets us repeat the tubular neighborhood theorem (see [9, section II.11] or [10, chapter 10]):

Theorem 1 (Tubular Neighborhood Theorem): If M is a compact, smoothly embedded submanifold, then there exists $\epsilon > 0$ such that $\mathrm{int}\, U_M^{\epsilon}$ is a tubular neighborhood of M.

It follows from the tubular neighborhood theorem that

$$r = \pi \circ \rho^{-1} : \rho\left(\mathrm{int}\, U_{\mathcal{N}M}^{\epsilon}\right) \to M \qquad (11)$$

is a smooth retraction of the tubular neighborhood onto M (again cf. [9, section II.11] or [10, chapter 10]). We define

$$Q : (\mathrm{int}\, U_M^{\epsilon})^n \to \mathbb{R}, \quad x \mapsto \sum_{i=1}^{n} \|x_i - r(x_i)\|^2. \qquad (12)$$

With the Lyapunov function $d(x, r^n(x))^2$ (where $r^n(x)$ is shorthand notation for $(r(x_1) \cdots r(x_n))$), one infers that M^n is an asymptotically stable invariant set of

$$\dot{x} = -\nabla Q(x), \qquad (13)$$

with any tubular neighborhood being its region of asymptotic stability, where ∇ denotes the usual gradient operator, i.e.,

$$\nabla Q(x) \cdot y = \lim_{h \to 0} \frac{Q(x + hy) - Q(x)}{h} \qquad (14)$$

for all $y \in \mathbb{R}^{nm}$. This is readily proven by noticing that $d(x, r^n(x))^2$ is differentiable within tubular neighborhoods, vanishing on M^n, and positive and regular elsewhere (for details on this, we refer to [11]).

We know that solutions to (6) locally approach B for initial conditions on M^n and that solutions to (13) approach M for initial conditions from tubular neighborhoods. Using the retraction r, we propose to combine the convergence properties of both differential equations to solve formation control problems. More particular, we propose the differential equation

$$\dot{x} = -\nabla Q(x) + \mathrm{grad}\, P(r^n(x)) =: X(x) \qquad (15)$$

to do so.

III. RETRACTION BALANCING AND FORMATION CONTROL

We propose the differential equation (15) in order to balance the retractions of $x_1 \cdots x_n$ and bring them to M at the same time, thus solving the formation control problem. Our main result is that B is an asymptotically stable set of equilibria of (15).

Theorem 2: B is an asymptotically stable set of equilibria of (15). Moreover, for every $\alpha \geq 0$ such that P is regular on $U_{\bar{P}}^{\alpha} \setminus B$, for every tubular neighborhood $\mathrm{int}\, U_M^{\epsilon}$ of M, for every $\delta < \epsilon$, $\{x \in (U_M^{\delta})^n | r^n(x) \in U_{\bar{P}}^{\alpha}\}$ is a subset of the region of asymptotic stability of B.

Proof: We separately prove that M^n is an asymptotically stable set of (15) and that

$$B_r = \{x \in \mathbb{R}^{nm} | P(r^n(x)) = p_0\} \qquad (16)$$

is an asymptotically stable set of (15) in order to conclude that $M^n \cap B_r = B$ is an asymptotically stable set of (15) by means of Lemma A1 (we postponed the lemma to the appendix).

We first show that M^n is an asymptotically stable set of (15). Therefore, consider the Lie derivative of Q along X given by

$$\mathrm{L}_X Q(x) = -\nabla Q(x) \cdot \nabla Q(x) + \nabla Q(x) \cdot \mathrm{grad}\, P(r(x)). \qquad (17)$$

Please note that this Lie derivative is only defined within tubular neighborhoods of M^n, as, by virtue of the tubular neighborhood theorem, smoothness of r can only be guaranteed within tubular neighborhoods.

We next show that $\nabla r^n(x)(x - r^n(x)) = 0$. To do so, remember that the tangent space of a regular level set of a continuously differentiable function at a point is contained in the nullspace of its gradient at this very point. Here, as M^n is the image of r^n, for all $y \in M^n$, for all $v \in \mathcal{T}_x(r^n)^{-1}(y)$ for a given $x \in (r^n)^{-1}(y)$, we have that $\nabla r^n(x) v = 0$. Now choose y to be $r^n(x)$. We then have that $(r^n)^{-1}(r^n(x)) = \{r^n(x)\} + \mathcal{N}_{r^n(x)} M^n$, an affine space, whose tangent space is thus $\mathcal{N}_{r^n(x)} M^n$. As, by its very definition, we have that $x - r^n(x) \in \mathcal{N}_{r^n(x)} M^n$, it thus follows that $\nabla r^n(x)(x - r^n(x)) = 0$. With this result at hand, as a consequence of the product rule, we find that the expressions containing ∇r^n in ∇Q vanish. It follows that

$$\nabla Q(x) = x - r^n(x), \qquad (18)$$

which is a member of $\mathcal{N}_{r^n(x)} M^n$.

However, by the very definition of the gradient operator (3), $\operatorname{grad} P\left(r^n\left(x\right)\right)$ must be in $\mathcal{T}_{r^n(x)} M^n$. Consequently, we have $\nabla Q\left(x\right) \cdot \operatorname{grad} P\left(r^n\left(x\right)\right) = 0$. It follows that

$$\mathrm{L}_X Q\left(x\right) = -\nabla Q\left(x\right) \cdot \nabla Q\left(x\right) \tag{19}$$

for all x in any tubular neighborhood of M^n. Now choose one such tubular neighborhood $(\operatorname{int} U_M^\epsilon)^n$ from any tubular neighborhood $\operatorname{int} U_M^\epsilon$ of M. Then $\mathrm{L}_X Q$ is negative on $(\operatorname{int} U_M^\epsilon)^n \setminus M^n$. As, moreover, Q is positive on the same set, it follows from Lyapunov's direct method that M^n is an asymptotically stable invariant set of (15). Now pick any $\delta < \epsilon$. As M is compact, $(U_M^\delta)^n$ will be a compact set on which $\mathrm{L}_X Q$ is nonnegative and negative only on M^n. As $(U_M^\delta)^n$ is a sublevel set of Q, it thus follows that $(U_M^\delta)^n$ is an invariant set of (15). Applying LaSalle's invariance principle, one finds that $(U_M^\delta)^n$ is a subset of the region of asymptotic stability of M^n.

Now prove asymptotic stability of B_r. Therefore, find that $r^n\left(x\right)$ is governed by the differential equation

$$\dot{r}^n\left(x\right) = \operatorname{grad} P\left(r^n\left(x\right)\right) =: R\left(r^n\left(x\right)\right) \tag{20}$$

again with the reason that $\nabla Q\left(r^n\left(x\right)\right)$ is in $\mathcal{N}_{r^n(x)} M^n$, whereas R must naturally be in $\mathcal{T}_{r^n(x)} M^n$. Now consider the Lie derivative of $\bar{P}$ along R given by

$$\mathrm{L}_R \bar{P}\left(r^n\left(x\right)\right) = \nabla \bar{P}\left(r^n\left(x\right)\right) \cdot \operatorname{grad} P\left(r^n\left(x\right)\right). \tag{21}$$

We have that $\nabla \bar{P} = -\nabla P$. As, furthermore, $\operatorname{grad} P$ is just the part of ∇P which is tangent to M^n, we have that

$$\mathrm{L}_R \bar{P}\left(r^n\left(x\right)\right) = -\operatorname{grad} P\left(r^n\left(x\right)\right) \cdot \operatorname{grad} P\left(r^n\left(x\right)\right). \tag{22}$$

It is a necessary condition for a point to be an extremum of a function that its gradient must vanish on this point. Moreover, the function must attain smaller values in a neighborhood of the extremum which is free of other critical points. Thus pick any $U_{\bar{P}}^\alpha$ which has this property. Then $\mathrm{L}_R \bar{P}$ is negative on $U_{\bar{P}}^\alpha \setminus B$. As, moreover, $\bar{P}$ is positive on $U_{\bar{P}}^\alpha \setminus B$ by its very definition, it follows from Lyapunov's direct method that B is an asymptotically stable invariant set of (20). Now, as $U_{\bar{P}}^\alpha$ is a sublevel set of $\bar{P}$ and $\mathrm{L}_R \bar{P}$ is nonpositive on $U_{\bar{P}}^\alpha$, $U_{\bar{P}}^\alpha$ is an invariant set of (20). As compactness of M implies compactness of $U_{\bar{P}}^\alpha$, it follows from LaSalle's invariance principle that $U_{\bar{P}}^\alpha$ is a subset of the region of asymptotic stability of B.

Now, as B is an asymptotically stable invariant set of (20) with $U_{\bar{P}}^\alpha$ being a subset of its region of asymptotic stability, it follows that B_r is an asymptotically stable invariant set of (15) with $\{x \in (U_M^\delta)^n | r^n\left(x\right) \in U_{\bar{P}}^\alpha\}$ being a subset of its region of asymptotic stability for any δ chosen as above. As $(U_M^\delta)^n$ was also a subset of the region of asymptotic stability of M^n, and as $M^n \cap B_r = B$, application of Lemma A1 and the fact that B is a set of equilibria of (15) conclude the proof. ∎

The theorem not only states that the differential equation (15) solves the formation control problem in the sense that B is asymptotically stable, but also that the region of asymptotic stability of B is "preserved" when compared to the region of asymptotic stability for the classical balancing algorithm.

IV. A Tutorial Example: The Circle

Some notions from our main result are rather abstract. Therefore, in this section, we explicitly compute some of these notions for tutorial purposes. Throughout this section, let $m = 2$ and let the formation M be the unit circle

$$M = \mathbb{S}^1 = \{x_i \in \mathbb{R}^2 | \, \|x_i\| = 1\}. \tag{23}$$

Further, consider $n = 5$ systems (i.e., $\mathscr{V} = \{1 \cdots 5\}$) such that M^5 becomes the 5-torus

$$M^5 = \mathbb{T}^5 = \{x \in \mathbb{R}^{10} | \, \|x_1\| = \cdots = \|x_5\| = 1\}. \tag{24}$$

The largest tubular neighborhood of $\mathbb{S}^1$ has radius 1 (the origin can not be uniquely retracted), i.e., $\operatorname{int} U_{\mathbb{S}^1}^1$ would be the largest tubular neighborhood we could possibly work with. Within this tubular neighborhood,

$$r : \operatorname{int} U_{\mathbb{S}^1}^1 \to \mathbb{S}^1, \, x_i \mapsto \frac{1}{\|x_i\|} x_i \tag{25}$$

defines a smooth retraction from the tubular neighborhood onto $\mathbb{S}^1$. The geodesic joining two points on $\mathbb{S}^1$, say $r\left(x_i\right)$ and $r\left(x_j\right)$, is conveniently defined via a rotation matrix, i.e.

$$\gamma_{r(x_i)}^{r(x_j)}\left(s\right) = R\left(\operatorname{acos}\left(r\left(x_i\right) \cdot r\left(x_j\right)\right) s\right) r\left(x_i\right), \tag{26}$$

where R is given by

$$R\left(\alpha\right) = \begin{bmatrix} \cos\left(\alpha\right) & -\sin\left(\alpha\right) \\ \sin\left(\alpha\right) & \cos\left(\alpha\right) \end{bmatrix}. \tag{27}$$

Evaluating the length functional, one obtains

$$\ell\left(\gamma_{r(x_i)}^{r(x_j)}\right) = \operatorname{acos}\left(r\left(x_i\right) \cdot r\left(x_j\right)\right). \tag{28}$$

With these notions at hand, the gradient of our potential function P is most conveniently computed using the chain rule, yielding

$$\sum_{j=1}^n w_{ij} \ell\left(\gamma_{r(x_i)}^{r(x_j)}\right) \frac{\mathrm{d}}{\mathrm{d}s} \gamma_{r(x_i)}^{r(x_j)}(s)\Big|_{s=0} \tag{29}$$

$$= \sum_{j=1}^n w_{ij} \operatorname{acos}\left(r\left(x_i\right) \cdot r\left(x_j\right)\right)^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} r\left(x_i\right) \tag{30}$$

$$= \sum_{j=1}^n w_{ij} \operatorname{acos}\left(\frac{1}{\|x_i\|} x_i \cdot \frac{1}{\|x_j\|} x_j\right)^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{1}{\|x_i\|} x_i \tag{31}$$

for the ith element of $\operatorname{grad} P\left(r^n\left(x\right)\right)$. On the other hand, with the identity (18), the ith element of $-\nabla Q\left(x\right)$ reads

$$r\left(x_i\right) - x_i = \left(\frac{1}{\|x_i\|} - 1\right) x_i, \tag{32}$$

bringing us into the position to recast (15) with notions from our example. In particular, the ith component of (15) reads

$$\dot{x}_i = \left(\frac{1}{\|x_i\|} - 1\right) x_i \tag{33}$$

$$+ \sum_{j=1}^n w_{ij} \operatorname{acos}\left(\frac{1}{\|x_i\|} x_i \cdot \frac{1}{\|x_j\|} x_j\right)^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{1}{\|x_i\|} x_i$$
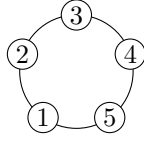
here.

Fig. 1. Cycle Graph $\mathscr{C}_5$ with unit weights (not indicated) and five nodes ($\widehat{i}$) such that the critical set B consists of equidistant points

We next turn our attention to compute an element of B and its corresponding elements of B_r. For doing so, we must first choose a communication graph. We choose the cycle graph $\mathscr{G} = \mathscr{C}_5$ with unit weights, i.e.

$$\mathscr{E} = \bigcup_{i \in \{1 \cdots 5\}} (i, (i \bmod 5) + 1) \cup ((i \bmod 5) + 1, i), \quad (34)$$

$$\mathscr{W} : (i, j) \mapsto 1 = w_{ij}, \quad (35)$$

which is depicted in Fig. 1. For this graph, the configurations in which the minimizing geodesics joining neighboring systems have length $2\pi/5$ are contained in B as π is the diameter of the manifold $\mathbb{S}^1$. As an example, consider the point $(0, 1)$. To obtain the corresponding balanced configuration, repeatedly rotate the point by $2\pi/5$, i.e.

$$\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, R\left(\frac{2\pi}{5}\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \cdots, R\left(\frac{2\pi}{5}\right)^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \in B. \quad (36)$$

This exemplary balanced configuration is depicted in Fig. 2. To obtain the corresponding element of B_r, one must compute the preimage of r^5 under this very element of B, which is most conveniently obtained by using the identity

$$\mathcal{N}_{r(x_i)} \mathbb{S}^1 = \operatorname{span}(x_i). \quad (37)$$

This reveals that

$$(r^5)^{-1} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, R\left(\frac{2\pi}{5}\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \cdots, R\left(\frac{2\pi}{5}\right)^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \quad (38)$$

$$= \left(\bigcup_{i \in \{0 \cdots 4\}} \operatorname{span}\left(R\left(\frac{2\pi}{5}\right)^i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \bigcap \operatorname{int} U_{\mathbb{S}^1}^1 \right) \subset B_r$$

i.e. the intersection of the union of vector spaces spanned by the balanced configuration with the tubular neighborhood. It is readily verified that the intersection of the latter with the 5-torus is again just the element of B which was initially constructed, i.e.,

$$\bigcup_{i \in \{0 \cdots 4\}} \operatorname{span}\left(R\left(\frac{2\pi}{5}\right)^i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \bigcap \mathbb{T}^5 \quad (39)$$

$$= \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, R\left(\frac{2\pi}{5}\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \cdots, R\left(\frac{2\pi}{5}\right)^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),$$

which was, simply said, the idea of the proof of Theorem 2. We consequently solved the differential equation (33) numerically in MATLAB using ode45 to illustrate the convergence properties of the differential equation. The solution to (33) is illustrated in Fig. 3, where $x_{i,0}$ denotes the initial condition, φ_i the solution, and φ_i^∞ the limiting value of φ_i, which is, for the chosen initial conditions, a balanced configuration.
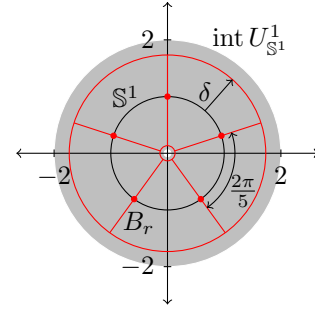

Fig. 2. An element of B (●) for the cycle graph from Fig. 1 consisting of equidistant points (the angular distance is $\frac{2\pi}{5}$) together with the corresponding set of points from $B_r \cap U_M^\delta$ (−), the boundary of U_M^δ (○) for some $\delta < \epsilon$, the tubular neighborhood int U_M^ϵ (●) for $\epsilon = 1$ (which is the largest possible tubular neighborhood in this example), all on the example of the unit sphere $M = \mathbb{S}^1$ (○)
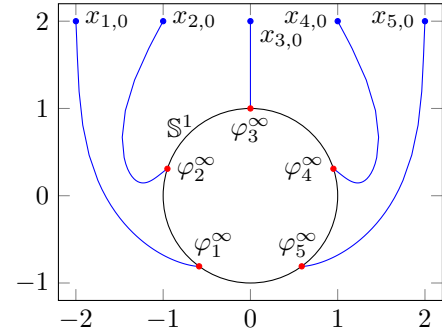

Fig. 3. Solution $\varphi = (\varphi_1 \cdots \varphi_5)$ (−) to the differential equation (15) together with its limiting value $\varphi^\infty = (\varphi_1^\infty \cdots \varphi_5^\infty)$ (●), which is contained in B from Fig. 2 for some initial condition $x_0 = (x_{1,0} \cdots x_{5,0})$ (●), all on the example of the unit sphere $M = \mathbb{S}^1$ (○) and the cycle graph from Fig. 1

V. A NUMERICAL EXAMPLE: THE TRIANGLE

Apart from the tutorial example from the foregoing section, as our algorithm is capable to deal with rather arbitrary formations via comparatively simple modifications, we study the case in which M is a triangle in this section. Triangular formations are relevant in robotics and aviation [13] but have also been subject to theoretical studies [14], [15]. We will let M be the "unit" triangle, i.e. the polygon with vertices

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, R\left(\frac{2\pi}{3}\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, R\left(\frac{2\pi}{3}\right)^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (40)$$

A minor issue here is that a polygon is no manifold, but this is easily coped with by locally smoothing out the corners of the polygon. In doing so, the triangle becomes a homotopy sphere and all computations from the foregoing section still apply. The question naturally arises for how a balanced configuration would look for the triangle. This is easily answered by computing the diameter of the manifold, which is given by

$$\frac{3}{2} \left\| R\left(\frac{2\pi}{3}\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| = \frac{3\sqrt{3}}{2}, \quad (41)$$

letting neighboring agents having Euclidean distance $3\sqrt{3}/5$ in a balanced configuration for $n = 5$ and $\mathscr{G} = \mathscr{C}_5$.
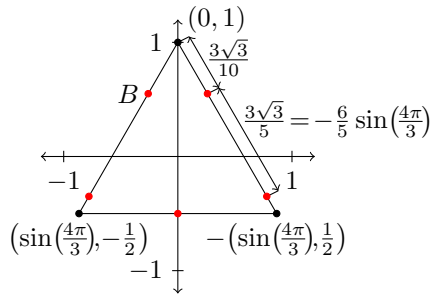
Fig. 4. An element of B (●) for the cycle graph from Fig. 1 consisting of equidistant points (the Euclidean distance is $3\sqrt{3}/5$) on the example of an equilateral triangle (–) with vertices (●) on the unit circle (we locally smoothed out the corners of the polygon in order to maintain the structure of a manifold)
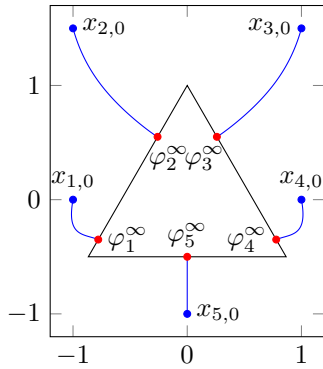

Fig. 5. Solution $\varphi = (\varphi_1 \cdots \varphi_5)$ (–) to the differential equation (15) together with its limiting value $\varphi^\infty = (\varphi_1^\infty \cdots \varphi_5^\infty)$ (●), which is contained in B from Fig. 4 for some initial condition $x_0 = (x_{1,0} \cdots x_{5,0})$ (●), all on the example of an equilateral triangle (–) and the cycle graph from Fig. 1

Algorithm 1 Construction of Balanced Configurations

Require: Vertices $V_1 \cdots V_p$ and point x_1 on polygon
1: **for all** $i = 1 \cdots p$ **do**
2: $\quad e_i \leftarrow e_{p+i} \leftarrow V_{(i \bmod p)+1} - V_i$
3: **end for**
4: $\mathtt{diam} \leftarrow \dfrac{1}{2}\displaystyle\sum_{i=1}^{p} \|e_i\|$
5: $e_0 \leftarrow 0$
6: **for all** $j = 1 \cdots n-1$ **do**
7: $\quad$ **for all** $i = 1 \cdots p$ **do**
8: $\quad\quad$ **if** $\exists s \in (0,1] : x_j = V_i + s e_i$ **then**
9: $\quad\quad\quad$ **for all** $k = p \cdots 0$ **do**
10: $\quad\quad\quad\quad$ **if**
$$\left\|x_j - V_{(i \bmod p)+1}\right\| + \sum_{l=1}^{k} \|e_{i+l}\| \geq \frac{2\mathtt{diam}}{n}$$
then
11: $\quad x_{j+1} \leftarrow x_j + \left(\dfrac{1}{\|e_i\|}e_i - \dfrac{1}{\|e_k\|}e_k\right)\left\|x_j - V_{(i \bmod p)+1}\right\|$
$$+ \sum_{l=1}^{k} e_{i+l}$$
$$+ \frac{1}{\|e_k\|}e_k\left(\frac{2\mathtt{diam}}{n} - \sum_{l=1}^{k}\|e_{i+l}\|\right)$$
12: $\quad\quad\quad\quad$ **end if**
13: $\quad\quad\quad$ **end for**
14: $\quad\quad$ **end if**
15: $\quad$ **end for**
16: **end for**
Return: $(x_1 \cdots x_n) \in B$

In order to construct such a balanced configuration, again pick an arbitrary point on M, say $(0, -1/2)$. Then, identify the vector of the edge of the polygon which it is located on. Here, this would be

$$\left(R\left(\frac{2\pi}{3}\right)^2 - R\left(\frac{2\pi}{3}\right)\right)\begin{bmatrix}0\\1\end{bmatrix}. \tag{42}$$

If the distance from the initial point $(0, -1/2)$ to the following vertex, which is here given by $\sqrt{3}/2$, is larger than or equal to $3\sqrt{3}/5$, then the next point is given by

$$\begin{bmatrix}0\\-\frac{1}{2}\end{bmatrix} + \frac{3}{5}\left(R\left(\frac{2\pi}{3}\right)^2 - R\left(\frac{2\pi}{3}\right)\right)\begin{bmatrix}0\\1\end{bmatrix}. \tag{43}$$

If this is not the case, then the next point is obtained from

$$\begin{bmatrix}0\\-\frac{1}{2}\end{bmatrix} + \frac{1}{2}\left(R\left(\frac{2\pi}{3}\right)^2 - R\left(\frac{2\pi}{3}\right)\right)\begin{bmatrix}0\\1\end{bmatrix}$$
$$+ \left(\frac{3}{5} - \frac{1}{2}\right)\left(\begin{bmatrix}0\\1\end{bmatrix} - R\left(\frac{2\pi}{3}\right)^2\begin{bmatrix}0\\1\end{bmatrix}\right), \tag{44}$$

whereby the last vector is merely the vector of the following edge. This procedure can be continued until obtaining the balanced configuration, which we depicted in Fig. 4.

The proposed procedure offers to compute balanced configurations for rather general polygons, which we demonstrate in Algorithm 1, following the procedure explained above iteratively. Therein, we assume a polygon with vertices $V_1 \cdots V_p$ and a point x_1 lying on the polygon given, to then iteratively compute the remaining elements of the balanced configuration. Firstly, the vectors of the edges $e_1 \cdots e_p$ are computed, followed by the computation of the diameter $\mathtt{diam}$ of the polygon. Next, it is determined on which edge the iterated point x_j lies. Consequently, it is checked on which edge the forthcoming point x_{j+1} will lie to thereafter assign it. This is done by adding the remaining length of the edge to the current point (unless the forthcoming point is on the same edge in which case k reduces to 0) to then also add the length coordinate on the edge on which the forthcoming point lies, plus all intermediate edges. The algorithm returns a balanced configuration.

Being in the position to compute elements of B, we also solved the differential equation (15) numerically in MATLAB using ode45 for this example to check whether solutions indeed approach B. The solution to (15) is illustrated in Fig. 5, where $x_{i,0}$ denotes the initial condition, φ_i the solution, and φ_i^∞ the limiting value of φ_i, which is, for the chosen initial conditions, a balanced configuration.

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VI. Conclusions

We presented a framework for formation control in which the shape of the formation is determined by a submanifold of the state space of the systems. We solved the formation control problem by simultaneously balancing the retractions of the systems from a tubular neighborhood onto the submanifold and asymptotically stabilizing the submanifold defining the formation. In doing so, we arrived at a distributed control law whose convergence properties were "inherited" from the classical balancing algorithm. The proposed framework admits for any compact smoothly embedded submanifold and we illustrated it on the examples of the unit circle and the "unit" triangle. In the former example, we computed all abstract notions from our main result explicitly, whereas in the latter example, we presented an algorithm to compute balanced configurations. Although we had to locally smoothen the corners of the triangle in order to maintain the structure of a manifold, we found that it is comparatively simple to adopt the algorithm to various submanifolds, particularly if they are homotopy spheres.

Appendix: An Auxiliary Result

The following lemma was employed in the course of the proof of Theorem 2.

Lemma A1: Let M_1, M_2 be asymptotically stable invariant sets with $M_1 \cap M_2$ nonempty. Then $M_1 \cap M_2$ is an asymptotically stable invariant set and its region of asymptotic stability is a superset of the intersection of the regions of asymptotic stability of M_1 and M_2.

Proof: Let $\varphi(x_0, \cdot)$ denote the solution to the differential equation under discussion initialized at x_0. Recall that stability of M_1 and M_2 means that we can choose any ϵ_1, $\epsilon_2 \in (0, \infty)$ and have existence of δ_1, $\delta_2 \in (0, \infty)$ such that

$$\forall x_0 \in U_{M_1}^{\delta_1}, \forall t \in [0, \infty), \varphi(x_0, t) \in U_{M_1}^{\epsilon_1}, \quad (45)$$

$$\forall x_0 \in U_{M_2}^{\delta_2}, \forall t \in [0, \infty), \varphi(x_0, t) \in U_{M_2}^{\epsilon_2}. \quad (46)$$

As $M_1 \cap M_2$ is nonempty, $U_{M_1}^{\delta_1} \cap U_{M_2}^{\delta_2}$ and $U_{M_1}^{\epsilon_1} \cap U_{M_2}^{\epsilon_2}$, respectively, are nonempty and neighborhoods of $M_1 \cap M_2$. We thus have

$$\forall x_0 \in U_{M_1}^{\delta_1} \cap U_{M_2}^{\delta_2} \forall t \in [0, \infty), \varphi(x_0, t) \in U_{M_1}^{\epsilon_1} \cap U_{M_2}^{\epsilon_2}. \quad (47)$$

Setting δ to $\min\{\delta_1, \delta_2\}$, with $U_{M_1 \cap M_2}^{\delta} \subset \left(U_{M_1}^{\delta_1} \cap U_{M_2}^{\delta_2} \right)$, this also yields

$$\forall x_0 \in U_{M_1 \cap M_2}^{\delta} \forall t \in [0, \infty), \varphi(x_0, t) \in U_{M_1}^{\epsilon_1} \cap U_{M_2}^{\epsilon_2}. \quad (48)$$

Moreover, as φ is continuous, we not only have that φ must remain in $U_{M_1}^{\epsilon_1} \cap U_{M_2}^{\epsilon_2}$, but also in the smallest isolated component of $U_{M_1}^{\epsilon_1} \cap U_{M_2}^{\epsilon_2}$ which contains $M_1 \cap M_2$, which we denote by C, and which is computed through

$$C_{12} = U_{M_1 \cap M_2}^{\epsilon_1} \cap M_2, \quad (49)$$

$$C_{21} = U_{M_1 \cap M_2}^{\epsilon_2} \cap M_1, \quad (50)$$

$$C = \left(U_{C_{21}}^{\epsilon_1} \cap U_{M_2}^{\epsilon_2} \right) \cap \left(U_{C_{12}}^{\epsilon_2} \cap U_{M_1}^{\epsilon_1} \right). \quad (51)$$

We then have that

$$\forall x_0 \in U_{M_1 \cap M_2}^{\delta} \forall t \in [0, \infty), \varphi(x_0, t) \in C. \quad (52)$$

Now suppose for contradiction that $M_1 \cap M_2$ is not stable, i.e. that

$$\exists \epsilon^\star \in (0, \infty) : \forall \delta \in (0, \infty), \exists x_0 \in U_{M_1 \cap M_2}^{\delta}, \exists t \in [0, \infty) :$$
$$\varphi(x_0, t) \notin U_{M_1 \cap M_2}^{\epsilon^\star}. \quad (53)$$

Reducing ϵ_1, ϵ_2 until

$$C \subset U_{M_1 \cap M_2}^{\epsilon^\star} \quad (54)$$

yields a contradiction, proving stability of $M_1 \cap M_2$, which was the first statement to be proven. It remains to prove attractivity of $M_1 \cap M_2$. This follows from the fact that the intersection of the regions of asymptotic stability of M_1 and M_2 is given by

$$\{x_0 \in \mathbb{R}^n | \lim_{t \to \infty} d(M_1, \varphi(x_0, t)) = 0 \text{ and}$$
$$\lim_{t \to \infty} d(M_2, \varphi(x_0, t)) = 0\} \quad (55)$$

and is thus contained in

$$\{x_0 \in \mathbb{R}^n | \lim_{t \to \infty} d(M_1 \cap M_2, \varphi(x_0, t)) = 0\}. \quad (56)$$

The fact that $M_1 \cap M_2$ is nonempty then implies that the intersection of the regions of asymptotic stability of M_1 and M_2 is a neighborhood of $M_1 \cap M_2$. This was the last statement to be proven. ∎

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