Formation Control via Rotation Symmetry Constraints

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***Abstract*— This work introduces a distributed formation control strategy for multi-agent systems based only on rotation symmetry constraints. We propose a potential function that enforces inter-agent rotational symmetries, with its gradient defining the control law driving the agents toward a desired symmetric and planar configuration. We show that only $(n-1)$ edges, the minimal connectivity requirement, are sufficient to implement the control strategy, where n is the number of agents. We further augment the design to address the maneuvering problem, enabling the formation to undergo coordinated translations, rotations, and scalings along a predefined virtual trajectory. Simulation examples are provided to validate the effectiveness of the proposed method.**

I. INTRODUCTION

The demand for distributed formation control schemes in multi-agent systems (MAS) has grown significantly in recent years, with applications ranging from UAV swarm coordination for mapping and surveillance [1], to satellite constellations coordination for efficient communication relays [2]. The role of a formation control law is to steer the agents into a desired spatial configuration in a distributed fashion. This is commonly achieved by imposing explicit geometric constraints between neighboring agents, such as distance-based schemes [3], [4], where inter-agent distances are fixed, or bearing-based schemes [5], where relative directions are maintained. In both cases, the desired target configuration is characterized using *only* local information shared between neighboring agents.

In many formations the desired configuration exhibits spatial symmetries between the agents - rotations and/or reflections, often inherent from sensing coverage or communication requirements. The work [6] introduced an approach leveraging formation symmetries together with inter-agent distance constraints, drastically reducing the required inter-agent communication links as compared to other approaches. This motivates the question of whether it is possible to design a formation control scheme that relies solely on symmetry constraints.

Graph theory provides the natural framework for modeling the decentralization, interaction topology, and geometric configuration of a MAS. Agents are represented as nodes (vertices), with their communication links as edges. A central challenge lies in balancing sparse information exchange while ensuring convergence to the desired configuration. To address this challenge, distance and bearing-based schemes leverage results from rigidity theory, relying on minimal infinitesimal rigidity (MIR) as a crucial architectural requirement to guarantee convergence to a desired shape [3], [5]. For distance-based approaches in $\mathbb{R}^2$, the MIR property requires at least $(2n-3)$ edges to uniquely determine the formation (up to translations, rotation, and so-called flip ambiguities), where n is the number of agents.

A relevant line of research builds on the concept of augmented Laplacian formulations such as complex-Laplacian approaches [7]–[9], where complex weights replace the standard scalar weights of the graph Laplacian, represent inter-agent interactions encoding rotations, translations, and scalings. Matrix-weighted Laplacian formulations, where the weights are structured matrices encoding inter-agent relations, offers another approach. This idea has appeared in several formation control settings: bearing-based control [5], where projection matrices enforce relative direction constraints; or more recent works [10], where the matrix weights play the same role as complex weights. These works demonstrate that matrix-weighted Laplacians form an ongoing field of research with multiple directions for achieving formation objectives.

In this work, we consider a group of n agents required to arrange themselves into a geometric spatial pattern specified *only* by a set of inter-agent rotation symmetry constraints. We formalize these constraints in Euclidean space as point group isometries corresponding to rotations. Point groups can be classified by several families [11]. In this paper we restrict our study to planar cyclic rotations, enforced between designated agent pairs. Each agent has access to its state in $\mathbb{R}^2$, and may exchange this information *only* with neighboring agents, as determined by an *undirected* interaction network. The control objective is to design a distributed control law that drives the agents from any initial configuration to a desired configuration satisfying the required rotation symmetry constraints between the agents.

This paper contributes with a foundation for solving the formation control problem solely under rotation constraints, providing a counterpart to the rigidity-based framework. We introduce a potential function that enforces rotational symmetries between neighboring agents, whose gradient yields a distributed control law driving the system toward the null-space of a symmetry-constraining matrix-weighted Laplacian. We show that $(n-1)$ edges — the minimal connectivity requirement — is sufficient to guarantee convergence to the desired formation. To enhance flexibility, we present an augmentation of the control strategy, enabling the desired formation to be achieved while undergoing coordinated translations, rotations, and scalings according to a

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time-varying reference virtual trajectory, effectively addressing the formation maneuvering problem. Additionally, the effectiveness of the approach is also demonstrated through a numerical example extension in $\mathbb{R}^3$.

The paper is organized as follows. Section II reviews the mathematics of symmetry, focusing on graphs and frameworks. Section III introduces the symmetry constrained formation control problem and presents the controller. Section IV extends the controller to allow for formation maneuvers, while Section V demonstrates numerically an extension to $\mathbb{R}^3$. Finally, concluding remarks are offered in Section VI.

Notations: A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of two non-empty sets: $\mathcal{V} = \{1, ..., n\}$ the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges. In this work, $\mathcal{G}$ is assumed to be undirected. The notation $ij \in \mathcal{E}$ indicates that agent i can receive information from its neighboring agent j, and vice versa. Let $I_n \in \mathbb{R}^{n\times n}$ be the identity matrix, and $\mathbb{1}_n \in \mathbb{R}^n$ be the all-one column vector of dimension n. Let $\otimes$ be the Kronecker product.

II. Symmetry in Graphs and Frameworks

The main focus of this work is to leverage the inherent symmetries of a formation to solve the formation control problem. In this direction, we first review notions from group theory and graph theory used to formally define symmetry.

A. Symmetry in Graphs

Group theory provides a powerful mathematical framework for describing symmetry. In the context of graphs, symmetries correspond to structure-preserving transformations of the vertex set—formally captured by the notion of *automorphisms*. The collection of all such transformations forms a group, known as the *automorphism group* of the graph. We begin by briefly recalling the definition of a group.

Definition 1. *A* group *is a set* Γ *equipped with a binary operation* $\circ$ *such that:*

- ***Closure:*** *For all* $a, b \in \Gamma$*, the composition* $a \circ b$ *is also in* Γ*.*
- ***Associativity:*** $(a \circ b) \circ c = a \circ (b \circ c)$ *for all* $a, b, c \in \Gamma$*.*
- ***Identity:*** *There exists an element* $\mathrm{id} \in \Gamma$ *such that* $a \circ \mathrm{id} = \mathrm{id} \circ a = a$ *for all* $a \in \Gamma$*.*
- ***Inverses:*** *For each* $a \in \Gamma$*, there exists an inverse* $a^{-1} \in \Gamma$ *such that* $a \circ a^{-1} = a^{-1} \circ a = \mathrm{id}$*.*

The order *of a group is the number of its elements. A subset* $B \subseteq \Gamma$ *that is itself a group under* $\circ$*, is called a* subgroup*.*

In the setting of graphs, these ideas appear naturally when considering automorphisms.

Definition 2. *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *be a finite, simple graph. An* automorphism *of* $\mathcal{G}$ *is a permutation* $\psi : \mathcal{V} \to \mathcal{V}$ *such that*

$$uv \in \mathcal{E} \quad \Leftrightarrow \quad \psi(u)\psi(v) \in \mathcal{E}.$$

That is, an automorphism preserves the adjacency structure of the graph. The identity permutation id is always an automorphism, and if ψ is an automorphism, then so is its inverse ψ^{-1}. Moreover, the composition of two automorphisms is again an automorphism. These properties ensure that the set of all automorphisms of $\mathcal{G}$ forms a group under composition. This group is called the *automorphism group* of $\mathcal{G}$, denoted by $\mathrm{Aut}(\mathcal{G})$. One can express every permutation as a composition of disjoint cycles of the permutation. A cycle is a successive action of the permutation that sends a vertex back to itself, i.e., $i \to \psi(i) \to \psi(\psi(i)) \to \cdots \to \psi^k(i) = i$, where $\psi^k = \underbrace{\psi \circ \cdots \circ \psi}_{k \text{ times}}$. Such a cycle is compactly written using the *cycle notation*, denoted by $(i\, \psi(i) \cdots \psi^{k-1}(i))$. The integer k is the *length* of the cycle.

Definition 3. *A graph* $\mathcal{G}$ *is* Γ-symmetric *for any subgroup* $\Gamma \subseteq \mathrm{Aut}(\mathcal{G})$*.*

Example 1. *Fig. 1 shows the cycle graph* C_3*. We can identify all the automorphisms of* $\mathrm{Aut}(C_3)$*. Naturally, we can directly identify the identity permutation* id*. Additionally, consider a counter-clockwise rotation by* $120°$ *of* C_3*. This gives the automorphism (in cycle notation)* $\psi_1 = (1\,2\,3)$*. We also have* $\psi_2 = \psi_1^2 = \psi_1 \circ \psi_1 = (1\,3\,2)$*, which can be interpreted geometrically as an additional rotation by* $240°$*. Additional permutations can be found by considering reflections. Consider first the reflection about the vertical blue line, giving the permutation* $\psi_3 = (1)(2\,3)$*. Similarly, the reflection about the red line yields* $\psi_4 = (3)(1\,2)$*, and the reflection about the green line gives* $\psi_5 = (2)(1\,3)$*. Thus, we have that* $\mathrm{Aut}(C_3) = \{\mathrm{id}, \psi_1, \ldots, \psi_5\}$ *has 6 automorphisms.*

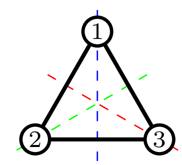


Fig. 1: Cycle graph C_3, with 6 automorphisms in $\mathrm{Aut}(\mathcal{G})$.

Note that we can choose $\Gamma = \{\mathrm{id}, \psi_1, \psi_1^2\}$, which corresponds to the subgroup of rotational automorphisms of C_3. In this case, C_3 can be considered as a Γ-symmetric graph, where any vertex can be mapped to any other under the rotation actions of Γ.

B. Symmetry in frameworks

The embedding of symmetric graphs in Euclidean space is of interest, especially for formation control problems. In this direction, we now consider symmetry of frameworks [12]. A framework in $\mathbb{R}^2$ is defined as the pair $(\mathcal{G}, p)$, where $p : \mathcal{V} \to \mathbb{R}^2$ assigns each node in $\mathcal{G}$ a position in Euclidean space, used to represent the physical position of the agents in the network.

Definition 4. *Let* Γ *be represented as a* point group*, i.e., a subgroup of the orthogonal group* $O(\mathbb{R}^2)$*, via a homomorphism* $\tau : \Gamma \to O(\mathbb{R}^2)$*, which assigns to each* $\gamma \in \Gamma$ *an isometry in* $\mathbb{R}^2$*. A framework* $(\mathcal{G}, p)$ *is called* $\tau(\Gamma)$-symmetric *if*

$$\tau(\gamma)p_i = p_{\gamma(i)} \quad \forall \gamma \in \Gamma, \quad i \in \mathcal{V}. \tag{1}$$

In this work, we restrict our study to frameworks whose underlying graph $\mathcal{G}$ is the cycle graph C_n. By using the standard Schoenflies notation for point groups [11], [13], we consider the rotational symmetries described by the cyclic point group $\mathcal{C}_n$ of order $n \geq 1$. That is, $\mathcal{C}_n$ specifies the rotation symmetries that map the agents into one another under rotations about the origin.

We define $\Gamma_r \in \mathrm{Aut}(C_n)$ to be the subgroup of rotational automorphisms of C_n, where each element $\tau(\gamma)$ is designed as a rotation about the origin by an angle $\theta = 2\pi/n$. Then, in a planar setting ($\mathbb{R}^2$), $\tau(\Gamma_r)$ coincides with the cyclic point group $\mathcal{C}_n$. We represent the elements $\tau(\gamma)$ by the standard rotation matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \in SO(2).$$

Thus, for any two vertices $u, v \in \mathcal{V}$ of a $\mathcal{C}_n$-symmetric framework, we denote by $\gamma_{uv} \in \Gamma_r$ the group element satisfying $\tau(\gamma_{uv})p_u = p_v$. Here, $\tau(\gamma_{uv})$ is represented by the rotation $R(\theta)$ and consequently, $\tau(\gamma_{vu}) = R(\theta)^T = R(-\theta)$. In each case, the desired configuration satisfies condition (1), where agent positions are mapped into one another by the respective rotation in $\mathcal{C}_n$.

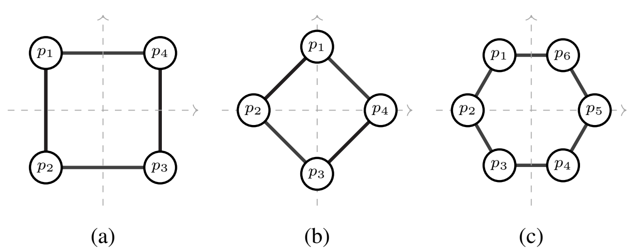


Fig. 2: Symmetric frameworks with C_n as the underlying graph. (a) and (b) are $\mathcal{C}_4$-symmetric, and (c) is $\mathcal{C}_6$-symmetric.

III. Symmetry-based Formation Control

We consider a team of n agents modeled by the integrator dynamics

$$\dot{p}_i(t) = u_i(t), \quad i \in \{1, \ldots, n\}, \tag{2}$$

where $p_i(t) \in \mathbb{R}^2$ is the position of agent i and $u_i(t) \in \mathbb{R}^2$ is its control to be designed. The coordination objective we consider is for the agents to arrange themselves into a configuration characterized by a specific symmetry class - rotation relationships between neighboring agents. Assume the desired configuration is a $\mathcal{C}_n$-symmetric framework with the cycle graph C_n as the underlying graph.

We define an *interaction graph* $\mathcal{G}_I = (\mathcal{V}, \mathcal{E}_I)$ to specify which agents are able to exchange information. This graph is defined as a spanning tree subgraph of C_n, to ensure the minimal connectivity requirement between the agents in the MAS. For example, for a $\mathcal{C}_4$-symmetric formation in Fig. 2(a), the interaction edge set $\mathcal{E}_I = \{(1,2),(2,3),(3,4)\}$ satisfies the connectivity requirement.

Let $\Gamma_r \subseteq \mathrm{Aut}(C_n)$ denote the subgroup of rotational automorphisms of C_n, and let $\mathcal{G}_I = (\mathcal{V}, \mathcal{E}_I)$ be the interaction graph defined as a spanning tree subgraph of C_n. The control objective is to design a distributed control law $u_i(t)$ for each agent such that, for every edge $uv \in \mathcal{E}_I$,

$$\lim_{t\to\infty} \left\| p_u(t) - \tau(\gamma_{vu})\, p_v(t) \right\| = 0. \tag{3}$$

Here, $\gamma_{vu} \in \Gamma_r$ is the permutation mapping v to u and $\tau(\gamma_{vu})$ is the associated point group element representing a rotation predefined for that edge.

We show that the interaction graph $\mathcal{G}_I$, chosen as a spanning tree subgraph of C_n, suffices to solve the formation control problem. This implies that only $(n-1)$ edges are required to guarantee convergence to a $\mathcal{C}_n$-symmetric formation.

A. Symmetry-based Control Law

Similar to the idea presented in [6], we define a *symmetry-forcing potential* over the edges in the interaction graph,

$$F(p(t)) = \frac{1}{2} \sum_{uv \in \mathcal{E}_I} \|p_u(t) - \tau(\gamma_{vu})p_v(t)\|^2. \tag{4}$$

To solve the formation control problem (3), we now propose the law defined by the gradient dynamical system

$$u(t) = -\nabla F(p(t)). \tag{5}$$

Then, we obtain the expression of the closed-loop dynamics for each agent i:

$$\dot{p}_i(t) = \sum_{ij \in \mathcal{E}_I} (\tau(\gamma_{ji})p_j(t) - p_i(t)). \tag{6}$$

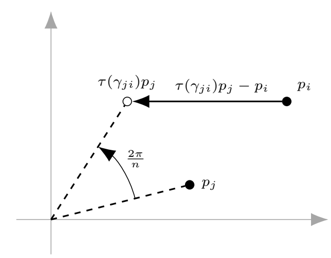


Fig. 3: The contribution $\tau(\gamma_{ji})p_j - p_i$ in the control law (6) drives p_i to a symmetric position of p_j.

The closed-loop dynamics of each agent (6) has a straight-forward geometric interpretation (see Fig. 3). The control law attempts to reduce the distance error between the term $\tau(\gamma_{ji})p_j$ and p_i.

We now focus on the closed-loop dynamics in state-space

$$\dot{p}(t) = -Qp(t), \tag{7}$$

$Q \in \mathbb{R}^{2n\times 2n}$ is the resulting *symmetry-constraining* matrix-weighted Laplacian for graph $\mathcal{G}_I$, with the block entries

$$[Q]_{uv} = \begin{cases} d(u)I_2, & u = v,\, u \in \mathcal{V} \\ -\tau(\gamma_{vu}), & vu \in \mathcal{E}_I \\ 0, & \text{o.w.} \end{cases},$$

where $d(u)$ denotes the degree of node u in the induced subgraph $\mathcal{G}_I$. As a matrix-weighted Laplacian, observe that

Q can be expressed as the matrix product $E(\Gamma_r)E(\Gamma_r)^T$, where $E(\Gamma_r) \in \mathbb{R}^{2n\times 2|\mathcal{E}_I|}$ has a matrix-weighted incidence matrix structure with its block-columns being associated with the edge ij,

$$[\cdots \underbrace{I_2}_{\text{node } i} \cdots \underbrace{-\tau(\gamma_{ji})^T}_{\text{node } j} \cdots]^T.$$

Example 2. *Consider the $\mathcal{C}_4$-symmetric framework seen in Fig. 2(a), with the choice of the edge set $\mathcal{E}_I = \{(1,2),(2,3),(3,4)\}$. Then, the group actions in $\tau(\Gamma_r)$ are rotations about the origin by $2\pi/n = \pi/2$, and the corresponding matrix $Q \in \mathbb{R}^{8\times 8}$ can be expressed as*

$$Q = \begin{bmatrix} I_2 & -R(\frac{\pi}{2})^T & 0 & 0 \\ -R(\frac{\pi}{2}) & 2I_2 & -R(\frac{\pi}{2})^T & 0 \\ 0 & -R(\frac{\pi}{2}) & 2I_2 & -R(\frac{\pi}{2})^T \\ 0 & 0 & -R(\frac{\pi}{2}) & I_2 \end{bmatrix}.$$

Note that $\text{Null}(Q)$ coincides with the set of $\mathcal{C}_4$-symmetric configurations satisfying (1). That is, $\tau(\gamma)p_i = p_{\gamma(i)}$ for all $\gamma \in \Gamma_r$ and $i \in \mathcal{V}$.

Proposition 1. *Let Q be the symmetry-constraining matrix-weighted Laplacian associated with the spanning tree graph $\mathcal{G}_I$. Then:*

i) Q is positive semi-definite (PSD).

ii) Q has a nontrivial null-space,

$$\text{Null}(Q) = \{p \in \mathbb{R}^{2n} | E(\Gamma_r)^T p = 0\},$$

corresponding to the set of $\mathcal{C}_n$-symmetric configurations.

iii) The rank of Q is $2n-2$, and $\dim \text{Null}(Q) = 2$.

Proof. (i) Since $Q = E(\Gamma_r)E(\Gamma_r)^T$, for any $p \in \mathbb{R}^{2n}$:

$$p^TQp = p^TE(\Gamma_r)E(\Gamma_r)^Tp = \|E(\Gamma_r)^Tp\| \geq 0.$$

Hence, Q is PSD.

(ii) Without loss of generality, assume the nodes are labeled such that each edge in $\mathcal{G}_I$ is of the form $i(i+1)$. By construction, $E(\Gamma_r)^Tp$ stacks the edge errors $r_i = p_i - \tau(\gamma_{(i(i+1))})^T\, p_{(i+1)} \in \mathbb{R}^2$, and

$$E(\Gamma_r)^\top p = \begin{bmatrix} r_1 \\ \vdots \\ r_{|\mathcal{E}_I|} \end{bmatrix} \in \mathbb{R}^{2|\mathcal{E}_I|}.$$

We have $E(\Gamma_r)^Tp = 0$ iff every edge satisfies $p_u - \tau(\gamma_{vu})p_v = 0$. Therefore, by Definition 4, $\text{Null}(Q)$ coincides with the respective set of $\mathcal{C}_n$-symmetric configurations.

(iii) Under the assumption that $\mathcal{G}_I$ is a spanning tree, for each node i define the matrix $S_i \in SO(2)$, with $S_1 = I_2$, and let

$$S_{i+1} = \tau(\gamma_{i(i+1)})S_i \in SO(2) \tag{8}$$

be the ordered product of edge rotations along the unique path from 1 to j. Define $p_1 = q \in \mathbb{R}^2$. Then, if $E(\Gamma_r)^Tp = 0$, by substitution we have

$$p_i \;=\; S_i\, q, \qquad \forall i \in \{1,\ldots,n\}.$$

such that

$$E(\Gamma_r)^T \begin{bmatrix} S_1q \\ S_2q \\ \vdots \\ S_nq \end{bmatrix} = 0.$$

Note that $q \in \mathbb{R}^2$ has two degrees of freedom.

Hence any vector in $\text{Null}(E(\Gamma_r)^T) = \text{Null}(Q) = \text{IM}(V_0)$ where

$$V_0 = \begin{bmatrix} S_1e_1 & S_1e_2 \\ S_2e_1 & S_2e_2 \\ \vdots & \vdots \\ S_ne_1 & S_ne_2 \end{bmatrix}, \tag{9}$$

with $e_1 = \begin{bmatrix}1 & 0\end{bmatrix}^T$ and $e_2 = \begin{bmatrix}0 & 1\end{bmatrix}^T$. Therefore, $\dim \text{Null}(Q) = 2$, and

$$\text{rank}(Q) = 2n - \dim \text{Null}(Q) = 2n-2.$$

□ □

We now examine the dynamics of the closed-loop system (7) to show that the proposed distributed control law drives the agents from any initial condition to the desired symmetric configuration. We next show the explicit solution of (7), showing that the limit configuration corresponds to the orthogonal projection of the initial state onto the subspace of $\mathcal{C}_n$-symmetric formations.

Theorem 1. *Consider a MAS consisting of n integrator agents (2), whose interaction topology is defined by a spanning tree graph $\mathcal{G}_I$, and let*

$$\mathcal{F} = \{p \in \mathbb{R}^{2n} | \tau(\gamma)p_i = p_{\gamma(i)},\;\; \forall \gamma \in \Gamma_r,\; i \in \mathcal{V}\}.$$

Then, for any initial condition $p(0) \in \mathbb{R}^{2n}$, the control (7) renders the set $\mathcal{F}$ exponentially stable, with $p(\infty)$ as the orthogonal projection of $p(0)$ onto $\mathcal{F}$,

$$\lim_{t\to\infty} p(t) = \frac{1}{n}V_0V_0^\top p(0), \tag{10}$$

where V_0 is given in (9).

Furthermore, the steady-state of each agent is given by

$$\lim_{t\to\infty} p_i(t) = \frac{1}{n}S_i\sum_{k=1}^{n} S_k^Tp_k(0), \tag{11}$$

with S_i defined as in (8).

Proof. By Proposition 1, note that V_0 is the vertical stack of the 2×2 blocks $S_i\begin{bmatrix}e_1 & e_2\end{bmatrix} = S_i$. Since $S_i \in SO(2)$ are orthogonal matrices we have

$$V_0^TV_0 = \sum_{i=1}^{n} S_i^TS_i = nI_2,$$

which shows that the columns of V_0 are orthogonal. Since Q is PSD and the columns of V_0 are orthogonal, we can define $\hat{V}$ as an orthonormal eigenbasis $\hat{V} = \begin{bmatrix}\hat{V}_0 & \hat{V}_+\end{bmatrix}$ with $\hat{V}_0 = \frac{1}{\sqrt{n}}V_0$ and V_+ the orthogonal complement of V_0 (in other words, V_+ corresponding to the non-zero eigenvalues $\Lambda_+ > 0$ of Q). Hence, Q is equivalent to

$$Q = \hat{V} \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_+ \end{bmatrix} \hat{V}^T,$$

and the closed-form solution of (7) results in

$$p(t) = e^{-Qt} p(0) = \hat{V} \begin{bmatrix} I_2 & 0 \\ 0 & e^{-\Lambda_+ t} \end{bmatrix} \hat{V}^T p(0).$$

Since all non-zero eigenvalues of $-Q$ are in OLHP, the dynamics of $p(t)$ exponentially converge to

$$\lim_{t \to \infty} p(t) = \hat{V}_0 \hat{V}_0^T p(0) = \frac{1}{n} V_0 V_0^\top p(0) \qquad (12)$$

Note that $V_0^T p(0) = \sum_{i=k}^{n} S_k^T p_k(0)$. Then, the block expression for each agent:

$$\lim_{t \to \infty} p_i(t) = \frac{1}{n} S_i \sum_{k=1}^{n} S_k^T p_k(0).$$

Moreover, observe that $E(\Gamma_r)^T p = 0$ implies $p_i = S_i q$ for some $q \in \mathbb{R}^2$. Hence, $\text{Null}(Q) = \text{IM}(V_0) = \mathcal{F}$, rendering $\mathcal{F}$ exponentially stable as claimed. □ □

Example 3. *Consider a MAS consisting of $n = 6$ agents, tasked with attaining a C_6-symmetric configuration (see Fig. 2(c)). Fig. 4 illustrates the underlying Γ-symmetric graph, with the dashed edge being removed for the chosen communication topology graph $\mathcal{G}_I$ with 5 edges. Note that by using a distance [3] or bearing [5] approach we would require 9 edges in total to ensure the correct formation shape in $\mathbb{R}^2$.*

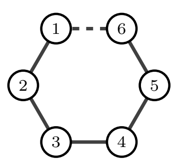


Fig. 4: Underlying graph $\mathcal{G}$.

Fig. 5 illustrates the trajectories and resulting configuration obtained by implementing the proposed control law (7), where the corresponding matrix Q is constructed using the rotation elements $\tau(\gamma_{uv}) = R(\pi/3)$.

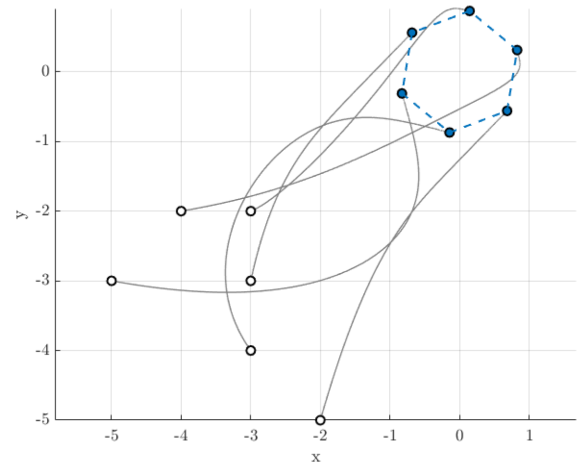


Fig. 5: Trajectories generated from (7).

IV. Formation Maneuvering

Our main focus has been on achieving and maintaining a target formation shape. Observe in Fig. 5 that due to symmetry the control (7) will successfully drive the agents to a desired formation shape but with respect to an inertial (global) origin. This may be limiting in many practical scenarios where the formation requires the ability to maneuver, that is, to translate, rotate, and scale while preserving the desired shape. To improve the flexibility of a symmetry-based formation control approach, [6] proposed augmenting the closed-loop dynamics for each agent (6) with a virtual state $r(t)$ that enables the agents to agree on a different origin. We leverage this idea to address formation maneuvering as well.

Assumption 1. *Assume each agent in the MAS has access to a virtual trajectory, predefined by*

i) a translation $r(t) \in \mathbb{R}^2$ with $\dot{r}(t) = v(t)$;
ii) a rotation $\mathcal{R}(t) \in SO(2)$ with $\dot{\mathcal{R}}(t) = \Omega(t)\mathcal{R}(t)$ where

$$\Omega(t) = \begin{bmatrix} 0 & -\omega(t) \\ \omega(t) & 0 \end{bmatrix},$$

and $\omega(t)$ is the desired angular velocity of the formation;
iii) a scale factor $s(t) \in \mathbb{R}^+$, with $\dot{s}(t) = \alpha(t)s(t)$, $\alpha(t) \in \mathbb{R}$.

Reference trajectories are known a priori in many applications [14]. Therefore, building on Assumption 1 we can define a shifted state for each agent, $c_i(t) = p_i(t) - r(t)$. Moreover, since the formation is specified with respect to the origin, we define the centroid of the formation at the origin and an axis of rotation passing through it. Under Assumption 1, we then propose the augmented control law

$$u(t) = -Qc(t) + \mathbb{1}_n \otimes v(t) + (I_n \otimes \Omega(t) + \alpha(t))c(t), \qquad (13)$$

for the agents to converge to the desired configuration while maneuvering along the predefined trajectory.

Theorem 2. *Consider a MAS consisting of n integrator agents (2) satisfying Assumption 1, whose interaction topology is defined by a spanning tree graph $\mathcal{G}_I$, and let*

$$\mathcal{F}_c = \{p \in \mathbb{R}^{2n} | \tau(\gamma)c_i = c_{\gamma(i)}, \ \ \forall \gamma \in \Gamma_r, \ i \in \mathcal{V}\}$$

be the set of all shifted C_n-symmetric configurations. Then, for any initial condition $p(0) \in \mathbb{R}^{2n}$, the control (13) renders the set $\mathcal{F}_c$ exponentially stable.

Proof. Define $\zeta(t) \in \mathbb{R}^{2n}$ to be the configuration $p(t) \in \mathbb{R}^{2n}$ expressed in a frame moving along the virtual trajectory,

$$\zeta(t) = \frac{1}{s(t)} \left(I_n \otimes \mathcal{R}(t)^T\right) c(t) \in \mathbb{R}^{2n}. \qquad (14)$$

We examine the derivative of each agent $\zeta_i(t) \in \mathbb{R}^2$ (product rule),

$$\dot{\zeta}_i(t) = -\frac{\dot{s}(t)}{s^2(t)}\mathcal{R}(t)^T c_i(t) + \frac{1}{s(t)}\big(\dot{\mathcal{R}}(t)^T c_i(t) + \mathcal{R}(t)^T \dot{c}_i(t)\big).$$

Note that $\dot{\mathcal{R}}(t)^T = -\mathcal{R}(t)^T\Omega(t)$. Since $\Omega(t)$ and $\mathcal{R}(t)$ commute in $\mathbb{R}^2$, we have $\dot{\mathcal{R}}(t)^T = -\Omega(t)\mathcal{R}(t)^T$. Then

$$\dot{\zeta}_i(t) = -\alpha(t)\zeta_i(t) - \Omega(t)\zeta_i(t) + \frac{1}{s(t)}\big(\mathcal{R}(t)^T(\dot{u}_i(t) - v(t))\big).$$

By applying the control (13), we have

$$\begin{aligned}\dot{\zeta}_i(t) &= -\alpha(t)\zeta_i(t) - \Omega(t)\zeta_i(t) - \frac{1}{s(t)}\mathcal{R}(t)^T v(t)\\ &+ \frac{1}{s(t)}\mathcal{R}^T\big(\sum_{ij\in\mathcal{E}_I}(\tau(\gamma_{ji})c_j(t) - c_i(t))\\ &+ v(t) + \Omega(t)c_i(t) + \alpha(t)c_i(t)\big).\end{aligned}$$

Since $\zeta_i(t) = \frac{1}{s(t)}\mathcal{R}(t)c_i(t)$, all the trajectory dependent terms cancel, simplifying the expression to

$$\dot{\zeta}_i(t) = \sum_{ij\in\mathcal{E}_I}(\tau(\gamma_{ji})\zeta_j(t) - \zeta_i(t)).$$

This reduces to the analysis of the agents $\dot{\zeta}(t) = -Q\zeta(t)$. By Theorem 1, the dynamics of $\zeta(t)$ ensure that the formation exponentially converges to the set

$$\mathcal{F}_\zeta = \{\zeta \in \mathbb{R}^{2n} | \tau(\gamma)\zeta_i = \zeta_{\gamma(i)},\ \ \forall\gamma \in \Gamma_r,\ i \in \mathcal{V}\}$$

From the definition of $\zeta_i(t)$ (14), this set is equivalent to $\mathcal{F}_c$, rendering the set $\mathcal{F}_c$ exponentially stable as claimed. □ □

Example 4. *Consider the same setup as in Example 3 under Assumption 1. A trajectory is predefined to enable the formation to maneuver through obstacles along a desired path. The blue line in Fig. 6 illustrates the trajectory state along the path, and the scaled arrows as the rotation state and scaling with respect to the initial state.*

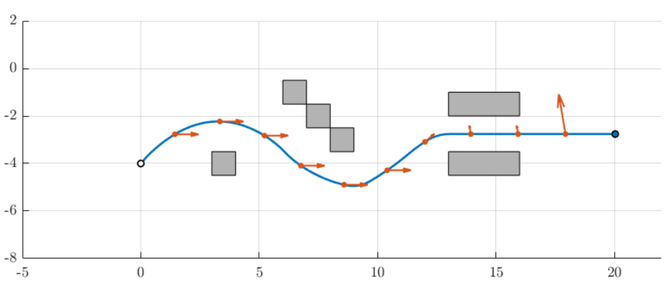


Fig. 6: Predefined reference trajectory implemented in control law (13).

Fig. 7 illustrates the resulting trajectories for each agent along the virtual trajectory predefined to each agent under control law (13).

To further evaluate the system, Fig. 8 illustrates the inter-agent rotation symmetry errors during maneuverings (3). *The errors exponentially converge to zero, which shows the effectiveness of the proposed method.*

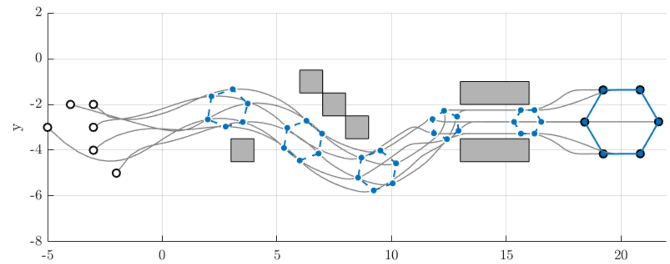


Fig. 7: Trajectories generated from (13) along the predefined trajectory.

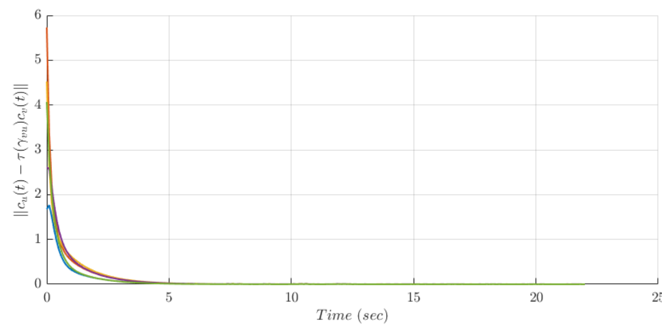


Fig. 8: Norm of the symmetry errors

V. Extension to $\mathbb{R}^3$

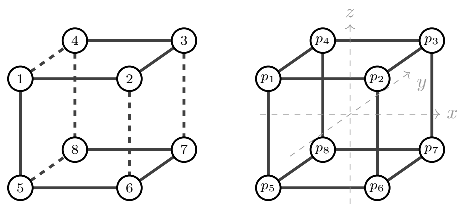
The results of Section IV are stated explicitly for formations in $\mathbb{R}^2$. Extending these ideas to $\mathbb{R}^3$ is relatively straightforward. In this section we provide a numerical example to illustrate how this might be done, with a formal analysis reserved for future work.

Consider a MAS consisting of $n = 8$ agents required to achieve a cube formation with an underlying graph $\mathcal{G}$ shown in Fig. 9(a). The target configuration can again be defined as a $\tau(\Gamma)$-symmetric framework, where the rotation elements $\tau(\Gamma)$ are now given by rotation matrices $R \in SO(3)$. In $\mathbb{R}^3$, such rotations are defined about coordinate axes, or general hyperplanes through the origin. For a cube formation, a natural choice of symmetries is given by $\mathcal{C}_4$ rotations about the coordinate axes (see Fig. 9(b)). For instance, the agents $\{p_1, p_2, p_3, p_4\}$ and $\{p_5, p_6, p_7, p_8\}$ may each satisfy a $\mathcal{C}_4$- symmetric framework about the z axis, and agents $\{p_2, p_1, p_5, p_6\}$ can be constrained to form a $\mathcal{C}_4$- symmetric framework orthogonal to z. These symmetry relations suffice for agents to exchange information according to a communication topology subgraph $\mathcal{G}_I$, obtained by removing the dashed edges in Fig. 9. By construction, $\mathcal{G}_I$ is a spanning tree. Similar to the planar case, we now define the symmetry-constraining matrix Q_z defined by $\mathcal{C}_4$ symmetries about the z axis, and $Q_\perp$ as the corresponding matrix defined by $\mathcal{C}_4$ symmetries about an orthogonal axis of z. The resulting symmetry-constraining matrix Q for the cube formation is then obtained as a composition

$$Q = I_2 \otimes Q_z + P\begin{bmatrix}Q_\perp^T & 0\end{bmatrix}^T P^T,$$

where P is a permutation matrix that reorders the block structure of $Q_\perp$ so that the composition of Q_z and $Q_\perp$

matches the indexing of the stacked state vector in the control law.



(a) (b)

Fig. 9: (a) The underlying graph $\mathcal{G}$, and (b) is the desired $\tau(\Gamma)$-symmetric framework of the cube formation.

By construction, $Q \succeq 0$, and its null-space corresponds to the set of cube-symmetric configurations. Hence the control (7) drives the system exponentially to the desired configuration.

Similar to the planar case, the method can be augmented with a virtual reference state $(r(t), R(t), s(t))$ to achieve maneuvering along a predefined trajectory. Fig. 10 shows the resulted trajectories by implementing control law (13). The agents converge to a cube formation along the predefined trajectory.

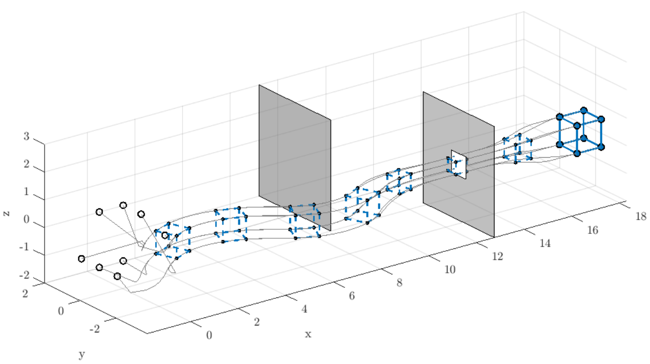


Fig. 10: Trajectories of the cube formation generated from (13) along a predefined trajectory.

VI. Concluding Remarks

We have introduced a formation control method to achieve a target formation by using only rotation symmetry constraints between neighboring agents. Notably, we have demonstrated that a communication spanning tree subgraph with $(n-1)$ constraints (matching the minimal connectivity requirement) suffices for the implementation of the protocol. By augmenting the control law with a time-varying virtual state we also proved the flexibility of the approach by enabling the ensemble to undergo translations, rotations, and scalings. In addition, we presented an extension of the method to achieve a formation in $\mathbb{R}^3$. This work demonstrates the potential of symmetry constraints in addressing formation control problems. Future directions include formally extending the framework to point group elements in $\mathbb{R}^3$, exploring directed, and switching graphs, and incorporating leader-follower configurations to generalize the approach to broader classes of configurations and enable fully distributed agreement on time-varying virtual trajectories.

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