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Maximum Hands-Off Distributed Bearing-Based Formation Control

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Abstract— **This paper investigates a distributed bearing-constrained formation control of continuous-time multi-agent systems based on sampled bearing information. The problem is considered in arbitrary dimensional spaces. Our proposed method penalizes the control effort by L^0 control cost, and hence the obtained distributed control is enhanced to take exactly zero value. Such a control is called maximum hands-off control. The proposed method tracks a distributed control for an associated discrete-time multi-agent system. Hence, we also newly characterize a bearing-constrained formation for discrete-time one. The analysis relies on the recently developed bearing rigidity theory. With the results, we show the feasibility, closed form, and stability of the proposed control.**

I. INTRODUCTION

Formation control is one of the cornerstone problems of multi-agent systems. Achieving a desired spatial configuration among autonomous agents may be an explicit requirement of the system [1], [2], or may be viewed as a subroutine for higher-level tasks [3], [4]. The standard setup for the formation control problem involves a team of dynamical systems, typically modeled as simple single- or double-integrators, that aim to distributedly achieve some desired spatial configuration specified by either relative positions, interagent distances, or interagent bearings [2], [5]. In this setting, there are numerous linear and nonlinear protocols that can be used to solve this problem.

While the existing control strategies for formation control do provide theoretical solutions to the problem, they often require significant modifications when implementing on real platforms. Challenges include adapting the control strategies for more complicated agent models, or coping with sensing and actuation constraints. A common approach to remedy this problem is to search for explicit nonlinear control strategies that address these issues [6]–[9].

In this work, we take the perspective that existing strategies based on simple models are advantageous to use even when implemented on more complicated systems. We can consider such a system as an exogenous *waypoint generator* or *trajectory planner*. This idea was originally proposed in [10], [11]. The previous studies tackled a consensus problem in which controllers have saturation constraints and can refer to only sampled-state information. For this problem, while they considered a continuous-time multi-agent system, they introduced an associated discrete-time multi-agent system as the trajectory planner and proposed a distributed consensus scheme satisfying the constraints.

For this work, we consider the bearing-based formation control problem originally presented in [12], [13]. This model assumes integrator dynamics for each agent with the ability to sense relative-positions to neighbors. The formation is specified by inter-agent bearings. It was shown in [12] that a linear protocol can solve this problem. In this work, however, we consider agents with integrator dynamics and input-saturation constraints. In addition, the model in this paper is assumed to be able to sense state information on only sampling instants. In order to address this complexity, this paper adopts the idea of a trajectory planner. As the planner, we design an appropriate discrete-time multi-agent system and analyze the convergence of the system. The obtained result newly characterizes the bearing-based formation control for a discrete-time system, compared to the previous studies for a continuous-time system [12], [13].

Furthermore, we try to reduce the control effort. In particular, we consider the L^0 cost minimization of the control input. This type of optimization enhances sparsity and is recently investigated across many fields, such as networked control [14], system identification [15], feedback design [16], to name a few. The control equipped with the minimum L^0 cost is called *sparse optimal control* or *maximum hands-off control* [17]. We adopt this optimal control motivated by the following three reasons.

Firstly, the maximum hands-off control has an ecological aspect. Indeed, this control strategy, which is also known as gliding or coasting, is actually used in hybrid/electric vehicles [18], railway vehicles [19], free-flying robots [20], stop-start systems [21], and sleep mode operation in wireless communication systems [22] for saving fuel or electricity consumptions and reducing CO_2 emissions or vibrations. Multi-agent systems whose energy sources are constrained may benefit from the sparse property of the maximum hands-off control. A similar strategy can be found in event-triggered (or self-triggered) control, which aims at increasing "zero control (or observation) time" to reduce computational/communication resources [23]. Secondly, the maximum hands-off control takes only discrete values, as shown in [24]. Such a discreteness is preferable for cheap actuators since the control values are already quantized. Thirdly, the maximum hands-off control is easily computed, since the control can be obtained by a convex program [24].

The organization of this paper is as follows. Section II provides mathematical preliminaries. Section III reviews the maximum hands-off control for readability. In Section IV, we first define the model in this paper and secondary explain

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the bearing-based formation control problem. Thirdly, we formulate the main problem, which tries to find a hands-off distributed formation control with input-saturation constraints based on sampled-state information. Section V is the main section of this paper. We propose a control algorithm that solves the main problem and show a condition under which a desired formation is achieved. Section VI illustrates the proposed control with a simulation. Section VII offers concluding remarks.

II. MATHEMATICAL PRELIMINARIES

This section reviews basic definitions and notations that will be used throughout the paper. The set of all real numbers is denoted by $\mathbb{R}$. The set of all positive integers is denoted by $\mathbb{N}$. For an integer $N \in \mathbb{N}$, $\mathbb{R}^N$ denotes the N-dimensional Euclidean space. We denote the vector in $\mathbb{R}^N$ whose elements are all 1 by 1_N. For two vectors z, $w \in \mathbb{R}^N$, the notation $z \le w$ corresponds to the component-wise inequality, i.e., $z \le w$ if $z_k \le w_k$ for all $k = 1, 2, \ldots, N$. The function $\max\{\cdot, \cdot\} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ returns the pointwise larger value, i.e., for $x = \max\{z, w\}$, then $x_k = \max\{z_k, w_k\}$ for each k. The function, $\min\{z, w\}$ is similarly defined. We denote $I_N \in \mathbb{R}^{N\times N}$ as the identity matrix and $\otimes$ as the Kronecker product. Let $\mathrm{Null}[M]$ be the null space and $\mathrm{rk}[M]$ be the rank of a matrix $M \in \mathbb{R}^{N\times N}$. The Euclidean norm in $\mathbb{R}^N$ is denoted by $\|\cdot\|$, i.e., $\|x\| \triangleq \sqrt{x^\top x}$ for $x \in \mathbb{R}^N$, where $\top$ denotes the transpose. The ℓ^p-norms in $\mathbb{R}^N$ with $p = 1$ or $p = \infty$ are respectively defined by

$$\|x\|_{\ell^1} \triangleq \sum_{m=1}^{N} |x_m|, \quad \|x\|_{\ell^\infty} \triangleq \max_{1\le m\le N} |x_m|$$

for $x = [x_1, x_2, \ldots, x_N]^\top \in \mathbb{R}^N$.

Let $T > 0$. For a continuous-time signal $u(t) = [u_1(t), u_2(t), \ldots, u_N(t)]^\top \in \mathbb{R}^N$ over a time interval $[0, T]$, we define its L^0 *and* L^∞ *norms* respectively by

$$\|u\|_0 \triangleq \sum_{m=1}^{N} \mu(\{t \in [0, T] : u_m(t) \neq 0\}),$$
$$\|u\|_\infty \triangleq \max_{m=1,2,\ldots,N} \operatorname*{ess\,sup}_{0\le t\le T} |u_m(t)|$$

with the Lebesgue measure μ on $\mathbb{R}$. The sign function is denoted by sgn, i.e., for a scalar $\alpha \in \mathbb{R}$, $\mathrm{sgn}(\alpha) \triangleq \alpha/|\alpha|$ if $\alpha \neq 0$, and $\mathrm{sgn}(0) \triangleq 0$.

An undirected graph, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, consists of a node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, where if information flows from a node $j \in \mathcal{V}$ to a node $i \in \mathcal{V}$, then $(j, i) \in \mathcal{E}$, and otherwise $(j, i) \notin \mathcal{E}$. The node $j \in \mathcal{V}$ is said to be a neighbor of a node $i \in \mathcal{V}$ if $(j, i) \in \mathcal{E}$, and the set of all neighbors of node i is denoted by $\mathcal{N}_i$, i.e., $\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. We denote by $|\mathcal{N}_i|$ the number of elements of the set $\mathcal{N}_i$. The maximum degree of a graph $\mathcal{G}$ is defined as $\max_{i\in\mathcal{V}} |\mathcal{N}_i|$ and we denote it by Δ.

III. MAXIMUM HANDS-OFF CONTROL

Here, we briefly review the maximum hands-off control proposed in [17]. Although we focus on first-order agents in this paper, we here discuss a more general linear time-invariant system modeled by

$$\dot{x}(t) = Fx(t) + Gu(t), \quad 0 \le t \le T_f, \tag{1}$$

where $F \in \mathbb{R}^{d\times d}$, $G \in \mathbb{R}^{d\times d}$, and $T_f > 0$ is a fixed final time of control. Note that the first-order system is modeled with $F = 0$ and $G = I_d$. For the system (1), we call a control u *feasible* if it steers the state x from a given initial state $x(0) = x_0 \in \mathbb{R}^d$ to a target state $x^f \in \mathbb{R}^d$ at time $T_f > 0$ (i.e., $x(T_f) = x^f$) and satisfies the magnitude constraint $\|u\|_\infty \le 1$. We denote by $\mathcal{U}(x_0, x^f, T_f)$ the set of all feasible controls for given $x_0 \in \mathbb{R}^d$, $x^f \in \mathbb{R}^d$, and $T_f > 0$.

The control objective in the maximum hands-off control is to obtain a control $u \in \mathcal{U}(x_0, x^f, T_f)$ that has the smallest support. In other words, this optimal control problem seeks the feasible control that has the minimum L^0 cost. This optimization problem will be solved by each agent in our proposed scheme, as described in Section V.

Problem 1 (Maximum Hands-Off Control): For a given initial state $x_0 \in \mathbb{R}^d$, target state $x^f \in \mathbb{R}^d$, and the final time of control $T_f > 0$, find a feasible control $u \in \mathcal{U}(x_0, x^f, T_f)$ that minimizes $J(u) \triangleq \|u\|_0$.

Solutions given by the maximum hands-off control lead to efficient control strategies (i.e., in terms of fuel or electricity consumption) since the actuator is inactive on the intervals where the control is exactly zero. Also, as described in Theorem 1 below, the maximum hands-off controls take only three values of ± 1 and 0. This property is called "bang-off-bang" property [25], and it is preferable for cheap actuators since the control values are already quantized.

The maximum hands-off control was recently investigated in [17], [24], [26]. Some fundamental properties of the optimal control is summarized below. This property will be used to show a sufficient condition for the feasibility of the proposed control scheme and a closed form of the proposed distributed control (see Theorem 4 and Lemma 1).

Theorem 1: Take any $T_f > 0$ and $x^f \in \mathbb{R}^d$. Define

$$\mathcal{R} \triangleq \left\{ \int_0^{T_f} e^{-Ft} G u(t) dt : \|u\|_\infty \le 1 \right\}. \tag{2}$$

Then, there exists at least one maximum hands-off control if and only if the initial state $x_0 \in \mathbb{R}^d$ satisfies $x_0 - e^{-FT_f} x^f \in \mathcal{R}$. Moreover, the hands-off controls are given by the L^1 optimal controls that take only values ± 1 and 0.

Proof: See the proof of [24, Theorem 3]. ■

IV. PROBLEM FORMULATION

In this paper, we adopt the idea of the maximum hands-off control to the bearing-based formation control problem in multi-agent systems. Let us consider a network model $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of n agents in $\mathbb{R}^d$ ($n \ge 2$, $d \ge 2$). We assume each agent $i \in \mathcal{V} \triangleq \{1, 2, \ldots, n\}$ is modeled by the single integrator dynamics,

$$\dot{x}_i(t) = u_i(t), \tag{3}$$

where $x_i(t) \in \mathbb{R}^d$ and $u_i(t) \in \mathbb{R}^d$ denote the state and the control of agent i, respectively. Note that $u_i(t)$ is a distributed control to be designed, which depends only on the i's state $x_i(t)$ and the states $x_j(t)$ of the neighboring agents $j \in \mathcal{N}_i$. The interaction graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is assumed to be connected, undirected, and fixed.

A. Bearing-Based Formation Control

The bearing-based formation control problem aims to drive a team of agents with dynamics (3) to a desired spatial configuration specified by inter-agent bearings. We briefly describe the approach presented in [12], [13].

For a given interaction graph $\mathcal{G}$, we define the *bearing* between two adjacent nodes as

$$g_{ij} = \frac{x_j - x_i}{\|x_i - x_j\|}.$$

Note that g_{ij} is a unit-length vector pointing from agent i to agent j. Furthermore, we assume bearings are measured in a common reference frame, and thus $g_{ij} = -g_{ji}$. We call the vector $x(t) = \begin{bmatrix} x_1^T(t) & \cdots & x_n^T(t) \end{bmatrix}^T$ the *configuration* of a team of agents, and $F_B(x(t)) = \left[g_1^T(t) \cdots g_{|\mathcal{E}|}^T(t) \right]^T \in \mathbb{R}^{nd}$ the *bearing function* associated with the configuration $x(t)$. We now state the bearing-based formation control problem.

Problem 2 (Bearing-based Formation Control): Given a set of feasible bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$, and a team of n agents with dynamics (3), design a distributed control u_i for each agent based on relative-position measurements $\{x_i(t) - x_j(t)\}_{(i,j)\in\mathcal{E}}$ such that $\lim_{t\to\infty} g_{ij}(t) = g_{ij}^*$ for all $(i,j) \in \mathcal{E}$.

Stated in another way, we would like $F_B(x(t)) \to F_B(x^*)$ for some configuration x^* that satisfies the desired bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$. Towards solving Problem 2, the notion of *infinitesimal bearing rigidity* is needed [5]. Simply stated, the pair $(\mathcal{G}, x(t_0))$ is infinitesimally bearing rigid if the only trajectories $x(t)$ that preserve the interagent bearings are the translations and scalings of the entire framework, i.e., $F_B(x(t))$ remains constant along these *trivial* trajectories.

The bearing rigidity of a framework can be checked by a rank-condition on the *bearing rigidity matrix*, defined as

$$R_B(x) = \frac{\partial F_B(x)}{\partial x}.$$

That is, the framework is infinitesimally bearing rigid if and only if $\mathrm{rk}[R_B(x)] = dn - d - 1$ [27]. An alternative matrix that can be used to check the bearing rigidity property is the *bearing Laplacian* matrix, $\mathcal{B}(x) \in \mathbb{R}^{dn\times dn}$ [13]. To define this matrix, we use the notion of an orthogonal projection matrix. We denote $P_{g_{ij}} \triangleq I_d - g_{ij}g_{ij}^T$ as the matrix operator projecting vectors onto the orthogonal complement of g_{ij}. With this notion, we define the bearing Laplacian as

$$[\mathcal{B}_{ij}(x)] = \begin{cases} 0_d, & i \neq j, (i,j) \notin \mathcal{E} \\ -P_{g_{ij}}, & i \neq j, (i,j) \in \mathcal{E} \\ \sum_{k\in\mathcal{N}_i} P_{g_{ik}}, & i = j, i \in \mathcal{V}. \end{cases} \quad (4)$$

Note that $\mathcal{B}(x)$ is a symmetric and positive semi-definite matrix for undirected graphs. For an infinitesimally rigid bearing configuration $(\mathcal{G}, x)$, we have $\mathrm{rk}[R_B(x)] = \mathrm{rk}[\mathcal{B}(x)]$ [13]. We often write $\mathcal{B}$ when the context is clear.

In this work we are concerned with a discrete-time formulation of the bearing-based formation control law which will serve as our trajectory planner of the system. In this direction, we consider the discrete-time single integrator dynamics for each agent,

$$z_i[k+1] = z_i[k] + w_i[k], \quad (5)$$

where $z_i[k] \in \mathbb{R}^d$ and $w_i[k] \in \mathbb{R}^d$ denote the state and the control of agent i at step $k \in \{0, 1, 2, \dots\}$, respectively. The proposed distributed control is defined by

$$w_i[k] \triangleq \varepsilon \sum_{j\in\mathcal{N}_i} P_{g_{ij}^*}(z_j[k] - z_i[k]), \quad (6)$$

where ε is a given positive number. This can be seen as a discretization of the continuous time controller originally proposed in [13].The state-space representation of the closed-loop system is thus

$$z[k+1] = (I_{nd} - \varepsilon\mathcal{B})z[k]. \quad (7)$$

Throughout the paper, we put the following assumption.

Assumption 1: Any configuration x^* that satisfies the bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ is infinitesimally bearing rigid.

Theorem 2: Under Assumption 1, any eigenvalue λ_l, $l = 1, 2, \dots, nd$, of the matrix $\mathcal{B}$ satisfies $0 \leq \lambda_l \leq 2d\Delta$.

Proof: The matrix $\mathcal{B}$ is positive-semi definite, as shown in [27]. Indeed, $\mathcal{B}$ can be written as $\mathcal{B}(x^*) = R_B^T(x^*)R_B(x^*)$, showing the lower eigenvalue bound. The upper bound follows directly from the Gershgorin circle theorem for block operator matrices [28] and the fact that the eigenvalues of projection matrices are either 0 or 1. ■

We now provide the main stability and convergence result for the system (7). In this direction, we first define the *formation centroid* (denoted $c(z)$), the *normalized target formation* (denoted r^*) as

$$c(z) \triangleq \frac{(1_n \otimes I_d)^\top z}{n}, \; r^* \triangleq z^* - 1_n \otimes c(z^*),$$

where z^* denotes a configuration satisfying the desired bearing constraints. It was shown in [12] that for infinitesimally bearing rigid configurations, $\mathrm{Null}[\mathcal{B}] = \mathrm{span}\{1_n \otimes I_d, r^*\}$.

Theorem 3: Let z^* be a configuration satisfying the bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ and $0 < \varepsilon < \frac{1}{d\Delta}$. Under Assumption 1, with the distributed control (6), the system (5) converges from any initial point $z[0]$ to a point

$$q^* = 1_n \otimes c(z[0]) + \left(\frac{r^{*\top}}{\|r^*\|} z[0] \right) \frac{r^*}{\|r^*\|}. \quad (8)$$

Proof: Under Assumption 1, the target formation is infinitesimally bearing rigid, and thus $\mathrm{rk}[\mathcal{B}] = dn - d - 1$. From Theorem 2, it follows that the spectrum of $A = I_d - \varepsilon\mathcal{B}$ is contained in the closed unit disc (i.e., A is semi-convergent). Let $Q = \begin{bmatrix} \frac{1}{\sqrt{n}} 1_n \otimes I_d & \frac{r^*}{\|r^*\|} \end{bmatrix}$. Then A is semi-convergent, and $\lim_{k\to\infty} A^k = QQ^\top$. It follows that $\lim_{k\to\infty} z[k] = q^*$ as required. ■

Remark 1: As discussed in [12], we require that $r^{*\top} z[0] > 0$ is additionally satisfied to solve Problem 2.

Otherwise, the system converges to a point reflection of the desired target formation (i.e., $g_{ij}[k] \to -g_{ij}^*$).

B. Main Problem

In this paper, we furthermore consider a *sampled-data* control system. The local interactions in multi-agent systems are usually performed on a wireless network and agents can transmit the state information at only the sampling times. Then, we assume that the state observation can be executed at only the sampling time kT, $k = 0, 1, 2, \ldots$, where $T > 0$ denotes the sampling period. As a result, we should determine the control $u_i(t)$, $t \in [kT, (k+1)T)$, based on relative-position measurements $\{x_i(kT)-x_j(kT)\}_{(i,j)\in\mathcal{E}}$. We also assume that the magnitude of the control is restricted to be bounded by 1, that is,

$$\|u_i\|_\infty \le 1, \quad \forall i \in \mathcal{V}. \tag{9}$$

In addition, it is important to take account of the control effort reduction for real systems. This paper adopts the maximum hands-off control idea given in the previous section to the sampled-data formation control. Let us denote the control u_i on the sampling interval $[kT, (k+1)T)$ by $u_i[k]$, that is,

$$u_i[k](t) \triangleq u_i(t + kT) \tag{10}$$

on $[0, T)$ for $i \in \mathcal{V}$. Then, this paper tries to find a distributed formation control $u_i[k]$ at every discrete time k that is equipped with small L^0 norm $\|u_i[k]\|_0$ and satisfies $\|u_i[k]\|_\infty \le 1$. In summary, we formulate the *maximum hands-off distributed bearing-based formation control problem* as follows:

Problem 3: Given feasible constant bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$, an initial state $x(0) \in \mathbb{R}^{nd}$, and a sampling period $T > 0$, find a control $u_i(t)$ on $[0, \infty)$ for each agent $i \in \mathcal{V}$ such that

i) it achieves $\lim_{t\to\infty} g_{ij}(t) = g_{ij}^*$ for all $\{i, j\} \in \mathcal{E}$,
ii) it satisfies the magnitude constraint (9),
iii) it is determined by sampled states $x_i(kT)$ and $x_j(kT) - x_i(kT)$ for $j \in \mathcal{N}_i$, $k = 0, 1, 2, \ldots$,
iv) and it minimizes $\|u_i[k]\|_0$ in a feasible set $\mathcal{U}(x_i(kT), x_i^f[k], T)$ with a given $x_i^f[k] \in \mathbb{R}^d$ at each $k = 0, 1, 2, \ldots$.

Note that our proposed control steers the state of each agent i to $x_i^f[k]$ on each sampling interval. Hence, the design of the target $x_i^f[k]$ is obviously crucial to the final formation. For this issue, we utilize a bearing-based formation for discrete-time multi-agent systems that is newly analyzed in Section IV and define $x_i^f[k]$ based on the result. In other words, our proposed algorithm tracks a distributed formation control for a corresponding discrete-time system.

V. MAXIMUM HANDS-OFF DISTRIBUTED CONTROL

In this section, we design a continuous-time distributed control that solves Problem 3. We first design a target state $x_i^f[k]$ based on obtained results and propose a distributed control, which we call *maximum hands-off distributed control*. We then analyze the feasibility of the proposed algorithm and the convergence of the multi-agent system.

A. Control Protocol

From the discussion above, a system defined by

$$z_i[k+1] = z_i[k] + \varepsilon \sum_{j\in\mathcal{N}_i} P_{g_{ij}^*}\big(z_j[k] - z_i[k]\big)$$

converges to the target formation as $k \to \infty$, under some conditions. Hence, if we choose a continuous-time control $u_i(t)$ that drives the state $x_i(t)$ from the current state $x_i(kT)$ to a target state

$$x_i^f[k] \triangleq x_i(kT) + \varepsilon \sum_{j\in\mathcal{N}_i} P_{g_{ij}^*}\big(x_j(kT) - x_i(kT)\big) \tag{11}$$

on each sampling interval $[kT, (k+1)T)$, then the agents (3) achieve a target formation at least on the sampling instants $t = 0, kT, 2kT, \ldots$. Note that the index $[k]$ in (11) corresponds to the current sampling instant kT, and $x_i^f[k]$ is the target state over the current sampling interval $[kT, (k+1)T)$. In particular, we are interested in a control that has the minimum L^0 cost $\|u_i[k]\|_0$ (i.e. the maximum hands-off control) among all feasible controls. Algorithm 1 shows the proposed control algorithm.

Algorithm 1 Maximum hands-off distributed control for agent $i \in \mathcal{V}$

Given a sampling period $T > 0$, an initial state $x_i(0) \in \mathbb{R}^d$, and a positive number $\varepsilon > 0$
for $k = 0, 1, 2, \ldots$ **do**
  Observe $x_i(kT)$ and $x_j(kT)$, $j \in \mathcal{N}_i$.
  Compute a maximum hands-off control

$$u_i[k] = \underset{u\in\mathcal{U}(x_i(kT), x_i^f[k], T)}{\arg\min} \|u\|_0$$

  where $x_i^f[k]$ is defined in (11).
  Apply $u_i(t) = u_i[k](t - kT),\ t \in [kT, (k+1)T]$ to the agent i.
end for

B. Analysis

The proposed control algorithm is analyzed in this subsection. In our framework, control inputs are constrained by the magnitude constraint (9), and they must steer agents to given target states over a finite horizon. Hence, we need to consider the feasibility of the algorithm, i.e., the existence of the control $u_i[k]$ in Algorithm 1 at each step k for all $i \in \mathcal{V}$. In addition, we need to show that the target formation is achievable in the continuous-time domain, since Theorem 3 refers to the convergence on the discrete sampling instants.

We first show the feasibility of Algorithm 1.

Theorem 4: Let us suppose Assumption 1. Fix an orthonormal system $\{q_1, q_2, \ldots, q_{nd}\}$ consisting of the eigenvectors of $\mathcal{B}$. Let $0 < \varepsilon < \frac{1}{d\Delta}$ and $T > 0$. Define a set

$$\mathcal{I} \triangleq \{x \in \mathbb{R}^{nd} : x = \sum_{i=1}^{nd} y_i q_i, \|y\|_{\ell^1} \le \frac{T}{2\varepsilon d\Delta}\},$$

where $y \triangleq [y_1, y_2, \ldots, y_{nd}]^\top$. If the initial state $x(0)$ satisfies $x(0) \in \mathcal{I}$, then there exists a maximum hands-off distributed control $u_i[k]$ in Algorithm 1 for all $i \in \mathcal{V}$ and $k = 0, 1, 2, \ldots$.

Proof: It follows from Theorem 1 that an agent $i \in \mathcal{V}$ has a maximum hands-off distributed control $u_i[k]$ at a step $k \in \{0, 1, 2, \ldots\}$ if and only if $x_i(kT) - x_i^f[k] \in \mathcal{R}$. Note that $\mathcal{R} = \{a \in \mathbb{R}^d : \|a\|_{\ell^\infty} \leq T\}$, which is obtained by (2) with $F = 0$ and $G = I_d$. Hence, it is enough to show the inequality $\|\varepsilon \mathcal{B} x(kT)\|_{\ell^\infty} \leq T$ for any $k = 0, 1, 2, \ldots$. This can be shown by mathematical induction. ■

Theorem 4 obviously depends on the selection of the basis $\{q_1, q_2, \ldots, q_{nd}\}$. Then, we here show a sufficient condition for the feasibility that can be more easily checked.

Corollary 1: Let us suppose Assumption 1. Fix any $0 < \varepsilon < \frac{1}{d\Delta}$ and $T > 0$. If the initial state $x(0)$ satisfies

$$\|x(0)\|_{\ell^1} \leq \frac{T}{2\varepsilon d\Delta\sqrt{nd}}, \tag{12}$$

then the maximum hands-off distributed control $u_i[k]$ exists for all $k \in \{0, 1, 2, \ldots\}$ and $i \in \mathcal{V}$.

Proof: Fix an initial state $x(0)$ such that (12) holds. Let us take any orthogonal matrix $Q \triangleq [q_1, q_2, \ldots, q_{nd}]$ consisting of eigenvectors of $\mathcal{B}$ such that Q diagonalizes $\mathcal{B}$. Then, there exists y such that $x(0) = Qy$, and $y = Q^\top x(0)$. Denote the m-th component of q_l and $x(0)$ by q_{lm} and $x^{(m)}(0)$, $m = 1, 2, \ldots, nd$. Then, we have

$$\|y\|_{\ell^1} \leq \sum_{l=1}^{nd} \sum_{m=1}^{nd} |q_{lm} x^{(m)}(0)| \leq \|x(0)\|_{\ell^1} \sqrt{nd},$$

where we used Cauchy-Schwartz inequality in the second inequality. Hence, it follows from (12) that $\|y\|_{\ell^1} \leq \frac{T}{2\varepsilon d\Delta}$, and $x(0) \in \mathcal{I}$, where $\mathcal{I}$ is defined in Theorem 4. This with Theorem 4 implies the existence of $u_i[k]$. ■

The following lemma gives the closed form of the maximum hands-off distributed control. Thanks to this characterization, the hands-off control can be easily obtained.

Lemma 1: Let us suppose Assumption 1. Fix an initial state $x(0)$ such that $x(0) \in \mathcal{I}$ for some orthonormal systems $\{q_1, q_2, \ldots, q_{nd}\}$ consisting of eigenvectors of $\mathcal{B}$, where $\mathcal{I}$ is defined in Theorem 4. Then, the set of all maximum hands-off distributed controls $u_i[k]$ in Algorithm 1 is given by

$$\mathcal{U}_i^*[k] \triangleq \Big\{ \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_d]^\top : \\ \alpha_m = -\beta_m \operatorname{sgn}(\theta_{im}[k]), \\ \beta_m(t) \in \{0, 1\} \text{ a.e. } t \in [0, T], \\ \|\beta_m\|_0 = |\theta_{im}[k]|, \quad \forall m = 1, 2, \ldots, d \Big\} \tag{13}$$

for all $k \in \{0, 1, 2, \ldots\}$ and $i \in \mathcal{V}$, where $\theta_{im}[k]$ is the m-th component of $x_i(kT) - x_i^f[k]$.

Proof: Note that, from Theorem 4, there certainly exists $u_i[k]$ for all $k = 0, 1, 2, \ldots$ and $i \in \mathcal{V}$. Since any maximum hands-off control is an L^1 optimal control that takes only values ± 1 and 0 from Theorem 1, we obtain the result by the discussion in [25, Sec. 8-2]. ■

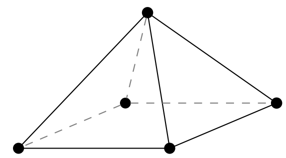

Fig. 1. Target formation.

Finally, we show that our proposed control in Algorithm 1 achieves the bearing-based formation control in the continuous-time domain.

Theorem 5: Let $0 < \varepsilon < \frac{1}{d\Delta}$ and $T > 0$. Fix an initial state $x(0)$ such that $x(0) \in \mathcal{I}$, which is defined in Theorem 4. Under Assumption 1, with the maximum hands-off distributed control in Algorithm 1, all agents converge to the point defined in (8), where $z[0]$ is replaced with $x(0)$, in the continuous-time domain.

Proof: Note that by Theorem 3 and Theorem 4 the maximum hands-off distributed control $u_i[k]$ exists at each sampling interval for each agent i and achieves a target formation at least on sampling instants. Denote the final formation by $x^* \in \mathbb{R}^{nd}$. Fix any $\eta > 0$. Then, there exists $K \in \mathbb{N}$ such that $k \geq K$ implies $\|x(kT) - x^*\| < \eta$, where $x(t)$ is the state trajectory corresponding to the controls $u_i[k]$. For any $t \in [kT, (k+1)T]$ with $k \geq K$, we have

$$\begin{aligned} \|x(t) - x^*\| &< \|x(t) - x(kT)\| + \eta \\ &\leq \|x((k+1)T) - x(kT)\| + \eta < 3\eta, \end{aligned}$$

where we used a fact that the state $x_i(t)$ lies between $x_i(kT)$ and $x_i((k+1)T)$ on each time interval $[kT, (k+1)T]$, i.e., $\underline{a}_i[k] \leq x_i(t) \leq \overline{a}_i[k]$ for all $t \in [kT, (k+1)T]$ and $i \in \mathcal{V}$, where $\underline{a}_i[k] \triangleq \min\{x_i(kT), x_i((k+1)T)\}$, $\overline{a}_i[k] \triangleq \max\{x_i(kT), x_i((k+1)T)\}$. This is observed from Lemma 1. Hence, $\|x(t) - x^*\| < 3\eta$ for $t \geq KT$. ■

VI. SIMULATION EXAMPLE

We demonstrate the hands-off control strategy with a simple example. Consider a network of 5 agents in $\mathbb{R}^3$. The agents should achieve a "pyramid" shape, as shown in Figure 1. Here, the target formation is specified by the bearings $g_{12}^* = \begin{bmatrix} -0.4851 & -0.7276 & -0.4851 \end{bmatrix}^T$, $g_{23}^* = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $g_{34}^* = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, $g_{41}^* = \begin{bmatrix} -0.4851 & 0.7276 & -0.4851 \end{bmatrix}^T$, $g_{13}^* = \begin{bmatrix} 0.4851 & -0.7276 & -0.4851 \end{bmatrix}^T$, $g_{15} = \begin{bmatrix} -0.4851 & -0.7276 & 0.4851 \end{bmatrix}^T$, $g_{52}^* = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T$, $g_{54}^* = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. Let the initial state be given by $x_1(0) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, $x_2(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, $x_3(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $x_4(0) = \begin{bmatrix} 0.5 & 0 & -0.5 \end{bmatrix}^T$, $x_5(0) = \begin{bmatrix} 0.5 & 0 & 0.5 \end{bmatrix}^T$, and take $\varepsilon = 0.01$, $T = 3$. Then, the feasibility of Algorithm 1 and the convergence of the multi-agent system are guaranteed from Corollary 1 and Theorem 5. In this setting, we simulated the maximum hands-off distributed bearing-based formation control.

Figure 2 shows the simulation result on the maximum hands-off distributed control $u_1 \triangleq [u_{11}, u_{12}, u_{13}]^\top$ and the corresponding state trajectory. We can confirm that the obtained control satisfies a magnitude constraint and is sufficiently sparse. Indeed, the L^0 cost of u_1 is only 0.633 on the interval $[0, 300]$. We can also see that the obtained distributed control guides the system to a target formation, and the final configuration is calculated as q^* in (8) from Theorem 5. For comparison, we also simulated the distributed bearing-based control proposed in [12]. The left of Fig. 3 shows the control v_1 on the interval $[0, 5]$ for agent 1. As we can see, the control takes values larger than 1 or smaller than -1. This is because the magnitude constraint was not considered in the formulation in [12]. In addition, the distributed bearing-based control $w_1[k]$ for the corresponding discrete-time system is shown in the right of Fig. 3. Since the sparsity of w_1 is not taken into account, the control w_1 is not sparse compared to the control u_1.

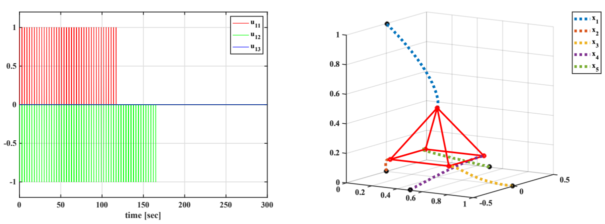

Fig. 2. Control input $u_1(t)$ (left) and the corresponding trajectory (right).

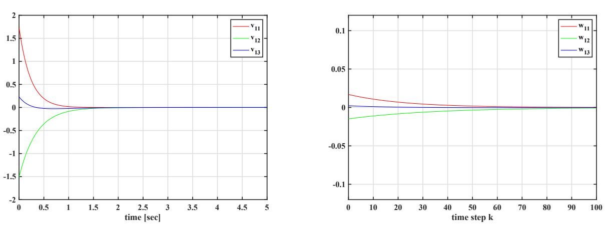

Fig. 3. Control inputs $v_1(t)$ (left) and $w_1[k]$ (right).

VII. Conclusions

This paper has considered a hands-off distributed control that achieves a bearing-constraint formation based on sampled-state observation. The proposed method uses an associated discrete-time multi-agent system as a trajectory planner, which suggests the target state on each sampling interval of each agent in a continuous-time system of interest. Then, we have analyzed the planner and have shown the convergence. Based on the results, we have shown the feasibility of the proposed algorithm and the convergence of the continuous-time system.

References

[1] L. Krick, M. E. Broucke, and B. A. Francis, "Stabilization of infinitesimally rigid formations of multi-robot networks," in *47th IEEE Conference on Decision and Control*, 2008, pp. 477–482.

[2] K.-K. Oh, M.-C. Park, and H.-S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, no. 0, pp. 424 – 440, 2015.

[3] X. Cai and M. de Queiroz, "Multi-agent formation maneuvering and target interception with double-integrator model," in *2014 American Control Conference*, June 2014, pp. 287–292.

[4] K. Oh and H. Ahn, "Formation control and network localization via orientation alignment," *IEEE Trans. Automat. Contr.*, vol. 59, no. 2, pp. 540–545, Feb 2014.

[5] S. Zhao and D. Zelazo, "Bearing rigidity and almost global bearing-only formation stabilization," *IEEE Trans. Automat. Contr.*, vol. 61, no. 6, pp. 1255–1268, July 2016.

[6] M. M. Asadi, A. Ajorlou, and A. G. Aghdam, "Distributed control of a network of single integrators with limited angular fields of view," *Automatica*, vol. 63, pp. 187–197, 2016.

[7] D. Dias, P. U. Lima, and A. Martinoli, "Distributed formation control of quadrotors under limited sensor field of view," in *International Conference on Autonomous Agents & Multiagent Systems*, 2016, pp. 1087–1095.

[8] Z. Sun, H. G. de Marina, G. S. Seyboth, B. D. O. Anderson, and C. Yu, "Circular formation control of multiple unicycle-type agents with nonidentical constant speeds," *IEEE Trans. Contr. Syst. Technol.*, vol. 27, no. 1, pp. 192–205, Jan 2019.

[9] J. P. Desai, J. P. Ostrowski, and V. Kumar, "Modeling and control of formations of nonholonomic mobile robots," *IEEE Trans. Robot. Automat.*, vol. 17, no. 6, pp. 905–908, Dec 2001.

[10] T. Ikeda, M. Nagahara, and K. Kashima, "Consensus by maximum hands-off distributed control with sampled-data state observation," in *55th IEEE Conference on Decision and Control*, 2016, pp. 962–966.

[11] ——, "Maximum hands-off distributed control for consensus of multi-agent systems with sampled-data state observation," *IEEE Trans. Control Netw. Syst.*, vol. 6, no. 2, pp. 852–862, 2019.

[12] S. Zhao and D. Zelazo, "Bearing-based distributed control and estimation of multi-agent systems," in *European Control Conference*, 2015, pp. 2202–2207.

[13] S. Zhao and D. Zelazo, "Translational and scaling formation maneuver control via a bearing-based approach," *IEEE Trans. Control Netw. Syst.*, vol. 4, no. 3, pp. 429–438, Sep. 2017.

[14] H. Fawzi, P. Tabuada, and S. Diggavi, "Secure estimation and control for cyber-physical systems under adversarial attacks," *IEEE Trans. Automat. Contr.*, vol. 59, no. 6, pp. 1454–1467, 2014.

[15] T. Chen, M. S. Andersen, L. Ljung, A. Chiuso, and G. Pillonetto, "System identification via sparse multiple kernel-based regularization using sequential convex optimization techniques," *IEEE Trans. Automat. Contr.*, vol. 59, no. 11, pp. 2933–2945, 2014.

[16] F. Lin, M. Fardad, and M. R. Jovanović, "Design of optimal sparse feedback gains via the alternating direction method of multipliers," *IEEE Trans. Automat. Contr.*, vol. 58, no. 9, pp. 2426–2431, 2013.

[17] M. Nagahara, D. E. Quevedo, and D. Nešić, "Maximum hands-off control: a paradigm of control effort minimization," *IEEE Trans. Automat. Contr.*, vol. 61, no. 4, 2016.

[18] C. Chan, "The state of the art of electric, hybrid, and fuel cell vehicles," *Proc. IEEE*, vol. 95, no. 4, pp. 704–718, 2007.

[19] E. Khmelnitsky, "On an optimal control problem of train operation," *IEEE Trans. Automat. Contr.*, vol. 45, no. 7, pp. 1257–1266, 2000.

[20] G. Vossen and H. Maurer, "On L^1-minimization in optimal control and applications to robotics," *Optimal Control Applications and Methods*, vol. 27, no. 6, pp. 301–321, 2006.

[21] B. Dunham, "Automatic on/off switching gives 10-percent gas saving," *Popular Science*, vol. 205, no. 4, p. 170, 1974.

[22] L. Kong, G. Wong, and D. Tsang, "Performance study and system optimization on sleep mode operation in IEEE 802.16e," *IEEE Trans. Wireless Commun.*, vol. 8, no. 9, pp. 4518–4528, 2009.

[23] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Trans. Automat. Contr.*, vol. 57, no. 5, pp. 1291–1297, 2012.

[24] T. Ikeda and K. Kashima, "On sparse optimal control for general linear systems," *IEEE Trans. Automat. Contr.*, vol. 64, no. 5, pp. 2077–2083, 2019.

[25] M. Athans and P. L. Falb, *Optimal Control*. Dover Publications, 1966.

[26] T. Ikeda and M. Nagahara, "Value function in maximum hands-off control for linear systems," *Automatica*, vol. 64, pp. 190–195, 2016.

[27] S. Zhao and D. Zelazo, "Bearing-only network localization: localizability, sensitivity, and distributed protocols," *Automatica*, vol. 69, pp. 334–341, 2016.

[28] C. Tretter, *Spectral Theory of Block Operator Matrices and Applications*. Imperial College Press, oct 2008. [Online]. Available: https://doi.org/10.1142/p493