Finite-Time Bearing-Only Formation Control via Distributed Global Orientation Estimation

Quoc Van Tran, Minh Hoang Trinh, Daniel Zelazo, Dwaipayan Mukherjee, and Hyo-Sung Ahn

***Abstract*—This paper presents a finite-time bearing-only formation control scheme via finite-time estimation of agents' global orientations. For the orientation estimation, we propose a distributed estimation law and establish almost global finite-time convergence. We provide analysis for undirected and a class of directed sensing and communication graphs. A distributed bearing-only formation control law is then proposed based on the orientation estimation. We provide a rigorous analysis for almost global stability and finite-time convergence of the system to the desired equilibrium. Particularly, under the proposed bearing-only formation control strategy, the actual formation almost globally exponentially finite-time converges to the desired formation shape. Finally, numerical simulations are provided to support our proposed control method.**

***Index Terms*—Bearing-only formation control, finite-time formation control, finite-time orientation estimation.**

I. Introduction

DISTRIBUTED formation control of multiagent systems has attracted much research attention in recent years [1], [2]. In distributed formation control of autonomous networked systems, the agents in a network aim to form a target formation shape by imposing kinematic or geometric constraints between neighboring agents. These constraints often characterize the variables to be sensed and controlled by the agents. Particularly, based on the constrained quantity, distributed formation control can be classified into displacement-based, distance-based, and bearing-based formation control [1].

Bearing-based formation control has been of particular interest in recent years partly due to the low cost and simplicity of the sensor systems associated with bearing measurements compared to distance-based or displacement-based solutions [3]–[11]. In the literature, bearing information can be interpreted as the subtended angle between two vectors (bearing angle) or the unit directional vector to a target (bearing vector). The bearing angle is a scalar quantity and, hence, it is invariant to any coordinate system.

As a result, the bearing-based formation control can be further categorized into two subgroups according to the control variable and whether a common reference coordinate frame is required. The first approach is controlling the bearing angles, and it requires no reference coordinate frame knowledge to each individual agent. For example, the early works on three- and four-agent formations can be found in [3] and [4], and an extension to more general n-agent systems is presented in [5]. A limitation of these works is that they are restricted to two-dimensional (2-D) spaces. The second method is controlling the bearing vectors to some neighboring agents of each agent in the network. The approach is developed based on the bearing rigidity theory in two or higher dimensions [6]–[8], and it requires that the agents in a networked system share the knowledge of the global coordinate system. A framework is characterized by an undirected sensing graph and the corresponding positions of agents in a Euclidean space. The bearing rigidity theory provides a powerful tool in examining the uniqueness of the formation shape and the condition for almost global convergence of bearing sensing frameworks in an arbitrary dimension [7]. There are applications in bearing-only formation control, network localization, and formation maneuvering control. For multiagent systems with directed sensing topologies, there is a work on the *leader-first follower* formation [9].

Without the knowledge of a common orientation, the bearing rigidity in *SE*(3) provides a distributed solution to the bearing-based formation control in 3-D [12]. This paper utilizes additional relative orientations among neighboring pairs. By using the relative orientation information, Zhao and Zelazo [7] introduced an orientation alignment scheme in SO(3) to the bearing-based formation control. By aligning agents' orientations and controlling interagent bearings simultaneously, the target formation can be achieved [7]. However, the orientation alignment method suffers from a strict restriction on the initial orientations [7], [13]. In SO(3), there are related works on orientation alignment, or attitude synchronization, [14], [15] and finite-time attitude synchronization for a leader–follower architecture [16], [17] and multiagent systems [18]. Instead of aligning agents' orientations, orientation estimation laws were proposed in [19]. In the orientation estimation scheme, agents' orientations are derived from controlled auxiliary variables defined in a vector

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space; thus, it guarantees almost global convergence of estimated orientations, and it is applicable in arbitrary dimensions.

This paper investigates the problem of *finite-time* formation stabilization for a group of autonomous agents by controlling interneighbor bearing directions, without the knowledge of a global coordinate frame. Finite-time controllers provide superior properties over those that ensure only asymptotic stability, such as faster convergence, better disturbance rejection, and robustness against uncertainties [20], [21]. Finite-time convergence, thus, is also often required in many applications where the control accuracy is crucial. Our recent work in [11] proposed two classes of finite-time control laws for the bearing-only formation control problem. For related works, there are results on finite-time consensus problems [22], [23], finite-time leader's state estimation, and tracking control of leader-following systems [24], [25], and finite-time distance-based formation control problems [5], [26]. However, a common sense of the global orientation is required for each agent in most of these works.[1]

A natural extension to these works is to consider the finite-time bearing-based formation control of multiagent systems via distributed estimation of the agents' orientations with regard to a reference coordinate system. For this, globally finite-time convergence of estimated orientations to the actual orientations is required. The first contribution of this paper, therefore, is the development of finite-time orientation estimation strategies. In particular, we propose an orientation estimation law by using relative orientations among interagent pairs for connected undirected and a class of directed relative orientation sensing graphs. By introducing an auxiliary matrix variable for each agent and analyzing the orientation estimation law in a linear space, we show that the estimated orientations almost globally exponentially converge to the actual orientations up to a constant coordinate rotation, in a finite time. We also derive upper bounds on the convergence time of the estimator. The finite-time orientation estimator is developed based on the orientation estimator proposed in [19] that ensures only asymptotic stability; furthermore, it can estimate time-varying orientations of agents, unlike [19].

Based on the proposed finite-time orientation estimation schemes, we investigate the finite-time bearing-only formation control without a common sense of the global coordinate frame. As the second contribution of this paper, a distributed bearing-only formation control law is proposed based on the orientation estimation. We first characterize the equilibrium set based on a set of modified desired bearings, which are transformed from the actual desired bearing vectors by a coordinate rotation. This control law is modified from the second control law[2] proposed in [11]. Complete analyses of almost global stability and finite-time convergence are provided in this paper, unlike [11], which is also the third contribution of this paper. Finally, simulation results are also provided to show the usefulness of the proposed method.

The rest of this paper is organized as follows. Preliminaries are given in Section II. The problem formulation is presented in Section III. A finite-time orientation estimation law is proposed in Section IV. Section V presents the proposed finite-time bearing-only formation control law and the main stability and convergence analyses . Finally, Section VI provides the simulations, and Section VII draws conclusions.

[1] Since distance is a coordinate-free quantity, distance-based formation control does not require a reference frame.

[2] Which uses the sign function.

II. Preliminaries

A. Notation

Let $\mathbf{x} = [x_1, \ldots, x_d]^T \in \mathbb{R}^d$ be a column vector in $\mathbb{R}^d$. We denote $|\mathbf{x}| = [|x_1|, \ldots, |x_d|]^T$. Let $\|\cdot\|$ be the 2-norm or Euclidean norm, $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$, and $\mathbf{1}_n = [1, \ldots, 1]^T \in \mathbb{R}^n$ denotes the vector of all ones. The $d \times d$ identity matrix is denoted by $\mathbf{I}_d$. For two matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \otimes \mathbf{B}$ denotes the Kronecker product between $\mathbf{A}$ and $\mathbf{B}$. The trace of a square matrix $\mathbf{A}$ is denoted by $\mathrm{tr}(\mathbf{A})$. Given $\mathbf{A}_i \in \mathbb{R}^{d\times d}$ for $i = 1, \ldots, m$, denote $\mathrm{diag}(\mathbf{A}_i) \triangleq \mathrm{blkdiag}(\mathbf{A}_1, \ldots, \mathbf{A}_m) \in \mathbb{R}^{dm\times dm}$.

For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ the Frobenius metric is given by $\|\mathbf{A} - \mathbf{B}\|_F = \sqrt{\mathrm{tr}\{(\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{B})\}}$, which is the Euclidean distance in $\mathbb{R}^{d\times d}$. The set of rotation matrices in $\mathbb{R}^d$ is denoted by $\mathrm{SO}(d) = \{\mathbf{Q} \in \mathbb{R}^{d\times d} | \mathbf{Q}\mathbf{Q}^T = \mathbf{I}_d,\ det(\mathbf{Q}) = 1\}$.

B. Finite-Time Convergence Theory

For $\mathbf{x} \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$; the function $\mathrm{sig}(\cdot)^\alpha : \mathbb{R}^d \to \mathbb{R}^d$ is defined as $\mathrm{sig}(x)^\alpha = [\mathrm{sign}(x_1)|x_1|^\alpha, \ldots, \mathrm{sign}(x_d)|x_d|^\alpha]^T$ [21]. The following inequality will be extensively used in this paper.

Lemma 1: [27] If $\xi_1, \ldots, \xi_d \geq 0$ and $0 \leq p \leq 1$, then

$$\left(\sum_{i=1}^{d} \xi_i\right)^p \leq \sum_{i=1}^{d} \xi_i^p.$$

A condition for finite-time convergence of continuous-time systems is given by the following lemma.

Lemma 2: ([21]). Suppose that there exists a positive-definite and continuous function $V(x,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$. If there exists $\kappa > 0$, $\alpha \in \{0, 1\}$, and open neighborhood $\mathcal{U}_0 \in \mathbb{R}^d$ of the origin such that

$$\dot{V} + \kappa V^\alpha \leq 0 \quad \forall \mathbf{x} \in \mathcal{U}_0 \setminus \{\mathbf{0}\}$$

then $V = 0$ for $t \geq T$, with the settling time $T \leq V^{1-\alpha}(0)/(\kappa(1-\alpha))$.

C. Graph Theory

An interaction graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is characterized by the vertex set $\mathcal{V} = \{1, \ldots, n\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge is defined by the ordered pair $e_k = (i, j)$. If $\mathcal{G}$ is undirected, $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$; j is a neighbor of i, and i is also a neighbor of j. If $\mathcal{G}$ is directed, $(i, j) \in \mathcal{E}$ is a directed edge from i to j, and this does not necessarily imply $(j, i) \in \mathcal{E}$. The set of neighbors of i is denoted by $\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ and the cardinality of $\mathcal{N}_i$ is denoted by $|\mathcal{N}_i|$. The Laplacian matrix, $\mathcal{L} = [l_{ij}]$, associated with $\mathcal{G}$, is defined as $l_{ij} = -1$ for $(i, j) \in \mathcal{E}$, $i \neq j$, $l_{ii} = -\sum_{j\in\mathcal{N}_i} l_{ij}\ \forall i \in \mathcal{V}$, and $l_{ij} = 0$ otherwise. For connected undirected graphs, $\mathrm{rank}(\mathcal{L}) = n - 1$ and its null space is given as $\mathcal{N}(\mathcal{L}) = \mathrm{span}(\mathbf{1}_n)$.

A directed path of a directed graph $\mathcal{G}$ is a sequence of edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$, where v_1 and v_k, respectively, are the start vertex and the end vertex. A directed path is a directed cycle if it has $v_1 \equiv v_k$. An acyclic digraph is a directed graph, which has no directed cycle. In this paper, we define a type of *rooted acyclic* directed graph (digraph) as follows.

Definition 1 (Rooted acyclic digraph): A *rooted acyclic* digraph is an acyclic directed graph constructed by the following sequence. There is a vertex 1 with no neighbor. Connect vertex 2 to 1 by the directed edge (2,1). Add 3 to $\{1, 2\}$ by one or two directed edges $(3, j)$, for some js $\in \{1, 2\}$. Similarly, a new vertex $i(3 \le i \le n)$ is added to $\{1, 2, \ldots, i-1\}$ by one or more directed edges (i, j), for some js $\in \{1, 2, \ldots, i-1\}$.

Obviously, a rooted acyclic digraph has no directed cycle, and node 1 is reachable by directed paths from all other nodes.

D. Bearing Rigidity Theory

A framework, denoted by $\mathcal{G}_b(\mathbf{p})$, is a mapping of the nodes in a graph $\mathcal{G}_b = (\mathcal{V}, \mathcal{E}_b)$ to a Euclidian space. In this paper, the undirected graph $\mathcal{G}_b$ represents the interaction graph used by the agents, and $\mathbf{p} = [\mathbf{p}_1^T, \ldots, \mathbf{p}_n^T]^T \in \mathbb{R}^{dn}$ denotes the configuration of the agents in the Euclidean space $\mathbb{R}^d$.

For each edge $(i, j) \in \mathcal{E}_b$, a corresponding displacement vector is defined as $\mathbf{z}_{ij} = \mathbf{p}_j - \mathbf{p}_i$. By fixing the order of the edges in $\mathcal{E}_b$, let $\mathbf{H} \in \mathbb{R}^{m \times n}$, where $m = |\mathcal{E}_b|$, denote the corresponding incidence matrix, then, the stacked displacement vector is defined as $\mathbf{z} = [\mathbf{z}_1^T, \ldots, \mathbf{z}_m^T]^T = (\mathbf{H} \otimes \mathbf{I}_d) = \bar{\mathbf{H}}\mathbf{p} \in \mathbb{R}^{dn}$.

Suppose that $\mathbf{p}_i \neq \mathbf{p}_j$. The bearing vector $\mathbf{g}_{ij}$ is the unit vector pointing along the direction of $\mathbf{z}_{ij}$, i.e.,

$$\mathbf{g}_{ij} = \frac{\mathbf{z}_{ij}}{\|\mathbf{z}_{ij}\|} = \frac{\mathbf{p}_j - \mathbf{p}_i}{\|\mathbf{p}_j - \mathbf{p}_i\|}. \tag{1}$$

The orthogonal projection operator corresponding to $\mathbf{g}_{ij}$ is defined by $\mathbf{P}_{\mathbf{g}_{ij}} = \mathbf{I}_d - \mathbf{g}_{ij}\mathbf{g}_{ij}^T$. It may be observed that $\mathbf{P}_{\mathbf{g}_{ij}}$ is symmetric, idempotent, and positive semidefinite, i.e., $\mathbf{P}_{\mathbf{g}_{ij}} = \mathbf{P}_{\mathbf{g}_{ij}}^2 = \mathbf{P}_{\mathbf{g}_{ij}}^T \geq 0$. Furthermore, $\mathbf{P}_{\mathbf{g}_{ij}}$ has eigenvalues $\{0, 1, \ldots, 1\}$, and its null space is given as $\mathcal{N}(\mathbf{P}_{\mathbf{g}_{ij}}) = \text{span}\{\mathbf{g}_{ij}\}$.

We denote the stacked bearing vector[3] $\mathbf{g} = [\mathbf{g}_1^T, \ldots, \mathbf{g}_m^T]^T \in \mathbb{R}^{dm}$ corresponding to the set of the ordered edges. The bearing rigidity matrix is defined as follows [7]:

$$\mathbf{R}(\mathbf{p}) = \frac{\partial \mathbf{g}}{\partial \mathbf{p}} = \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\frac{\partial \mathbf{z}}{\partial \mathbf{p}} = \text{diag}\left(\frac{\mathbf{P}_{\mathbf{g}_k}}{\|\mathbf{z}_k\|}\right)\bar{\mathbf{H}}. \tag{2}$$

Note that $\mathbf{R}(\mathbf{p}) \in \mathbb{R}^{dm \times dn}$. Furthermore, for any bearing rigidity matrix, $\text{span}\{\text{Range}(\mathbf{1}_n \otimes \mathbf{I}_d), \mathbf{p}\} = \text{span}\{\text{Range}(\mathbf{1}_n \otimes \mathbf{I}_d), \mathbf{p} - \mathbf{1}_n \otimes \bar{\mathbf{p}}\} \subseteq \mathcal{N}(\mathbf{R}(\mathbf{p}))$, where $\bar{\mathbf{p}} = \frac{1}{n}\sum_{i=1}^n \mathbf{p}_i$ is the formation centroid. Consequently, $\text{rank}(\mathbf{R}(\mathbf{p})) \leq dn - d - 1$. A framework $\mathcal{G}_b(\mathbf{p})$ is said to be infinitesimally bearing rigid if and only if $\text{rank}(\mathbf{R}(\mathbf{p})) = dn - d - 1$ [7]. Moreover, an infinitesimally bearing rigid framework can be uniquely determined up to a translation and a scaling factor. We borrow the following lemma from [7].

[3]Although both $\mathbf{g}_{ij}$ and $\mathbf{g}_k$ are used to denote bearing vectors, the notation will be clear from the context in each part of this paper.

Lemma 3: [7] A framework $\mathcal{G}_b(\mathbf{p})$ is infinitesimally bearing rigid if and only if $\mathcal{N}(\mathbf{R}(\mathbf{p})) = \text{span}\{\text{Range}(\mathbf{1}_n \otimes \mathbf{I}_d), \mathbf{p} - \mathbf{1}_n \otimes \bar{\mathbf{p}}\}$, where $\bar{\mathbf{p}} = (1/n)\sum_{i=1}^n \mathbf{p}_i = (1/n)(\mathbf{1}_n \otimes \mathbf{I}_d)^T\mathbf{p}$.

E. Gram–Schmidt Orthonormalization Procedure (GSOP)

For a set of d independent vectors $\mathcal{Z} = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_d\}$ in $\mathbb{R}^d$, the GSOP, which constructs d orthonormal column vectors of $\mathbf{Q} \in \text{SO}(d)$ from $\mathcal{Z}$, is defined as follows:

$$\begin{aligned}
\mathbf{v}_1 &:= \mathbf{z}_1 & \mathbf{q}_1 &:= \mathbf{v}_1/\|\mathbf{v}_1\| \\
\mathbf{v}_2 &:= \mathbf{z}_2 - \langle \mathbf{z}_2, \mathbf{q}_1 \rangle \mathbf{q}_1 & \mathbf{q}_2 &:= \mathbf{v}_2/\|\mathbf{v}_2\| \\
&\vdots & &\vdots \\
\mathbf{v}_d &:= \mathbf{z}_d - \sum_{k=1}^{d-1} \langle \mathbf{z}_d, \mathbf{q}_k \rangle \mathbf{q}_k & \mathbf{q}_d &:= \alpha \mathbf{v}_d/\|\mathbf{v}_d\|
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product, and the coefficient α is chosen such that $\det(\mathbf{Q}) = 1$ as $\alpha := \text{sign}(\det([\mathbf{q}_1, \ldots, \mathbf{q}_{d-1}, \mathbf{v}_d/\|\mathbf{v}_d\|]))$.

If the set $\mathcal{Z}$ contains a linearly dependent vector (which linearly depends on one or more vectors in $\mathcal{Z}$), there exists $\mathbf{v}_i = \mathbf{0}$, $i \in \{1, \ldots, d\}$, and hence $\mathbf{q}_i = \mathbf{0}$.

III. Problem Formulation

Consider a system of n autonomous agents in $\mathbb{R}^d$, $d = \{2, 3\}$. The agents do not have information about a common reference frame; each agent maintains a local coordinate frame, which fixes to the agent. The dynamics of an agent i are characterized by the single-integrator dynamics

$$\dot{\mathbf{p}}_i^i = \mathbf{u}_i^i \tag{3}$$

where $\mathbf{p}_i^i, \mathbf{u}_i^i \in \mathbb{R}^d$, respectively, are the position and control inputs of agent i, expressed in its body-fixed coordinated frame, i.e., ${}^i\Sigma$. Note that the single-integrator dynamics (3) is widely used in navigation and formation control literature [1]. Furthermore, the single-integrator is often considered first and can be generalized to adapt for more complicated models and constraints [10]. Although there have been some existing works on formation control of more practical agent models [10], [28], either the global state of the agent or a global coordinate frame is required. Extensions of this work for agents with more general dynamics will be addressed in our future papers.

The (global) orientation, or attitude, of agent i in $\mathbb{R}^d$ can be characterized by a square, orthogonal matrix $\mathbf{Q}_i \in \text{SO}(d)$ whose column vectors represent the coordinates of the orthogonal bases of the ith local coordinate frame expressed in the global coordinate frame. Thus, $\mathbf{Q}_i$ can be understood as the rotation matrix, which rotates the global coordinate system denoted by Σ, to the local coordinate frame denoted by ${}^i\Sigma$. Note that a rotation matrix in $\text{SO}(d)$ is unique and globally defined [29]. Let $\mathbf{Q}_k \in \text{SO}(d)$ be the orientation of agent $k\,\forall k \in \{1, \ldots, n\}$. Then, the relative orientation of the jth local coordinate frame with regard to the ith local coordinate frame is defined by

$$\mathbf{Q}_{ij} = \mathbf{Q}_i^{-1}\mathbf{Q}_j = \mathbf{Q}_i^T\mathbf{Q}_j. \tag{4}$$

The agents in the system aim to form a desired/target formation shape without the knowledge of a global orientation. The desired formation is characterized by a feasible set of bearing vectors $\{\mathbf{g}_{ij}^*\}_{(i,j)\in\mathcal{E}_b}$ expressed in Σ. Suppose that there exists an infinitesimally bearing rigid framework, $\mathcal{G}_b(\mathbf{p}^*)$, realized from the set $\{\mathbf{g}_{ij}^*\}_{(i,j)\in\mathcal{E}_b}$. Note that for a constant coordinate rotation $\mathbf{Q}\in\mathrm{SO}(d)$, the objective of the formation control is equivalent to achieve the desired formation specified by the bearing set $\{\mathbf{Q}\mathbf{g}_{ij}^*\}_{(i,j)\in\mathcal{E}_b}$. Furthermore, the n-agent system satisfies the following assumptions.

Assumption 1: Each agent in the system can sense the relative orientations with regard to one or more neighboring agents. Furthermore, the relative orientation sensing topology is given by an interaction graph $\mathcal{G}_o$ (directed or undirected).

Assumption 2: Each agent in the system can sense the bearing vectors with regard to several neighboring agents, and the bearing sensing topology is specified by a fixed undirected graph $\mathcal{G}_b$.

The (relative) orientation sensing topology is characterized by the graph $\mathcal{G}_o=(\mathcal{V},\mathcal{E}_o)$. Note that $\mathcal{V}=\{1,\ldots,n\}$ and $\mathcal{E}_o\subseteq\mathcal{V}\times\mathcal{V}$. Specifically, if $(i,j)\in\mathcal{E}_o$, agent i senses the relative orientation $\mathbf{Q}_{ij}$ and estimates its orientation, which is denoted as $\hat{\mathbf{Q}}_i\in\mathrm{SO}(d)$, based on the relative orientation information $\mathbf{Q}_{ik},\ k\in\mathcal{N}_i$. Furthermore, if $(i,j)\in\mathcal{E}_o$, agent j communicates an auxiliary matrix (which will be defined in the following section) to i. Thus, the graph $\mathcal{G}_o$ is both a sensing and communication network. Let $\mathcal{L}_o$ be the Laplacian of $\mathcal{G}_o$.

Until now, we have defined two distinct interaction graphs associated with the orientation estimation and formation control tasks, i.e., $\mathcal{G}_o$ and $\mathcal{G}_b$, respectively. In this paper, $\mathcal{G}_o$ can be undirected or directed. When $\mathcal{G}_o$ is undirected, $\mathcal{G}_o$ and $\mathcal{G}_b$ may be identical, i.e., $\mathcal{E}_o=\mathcal{E}_b$, or different (i.e., $\mathcal{E}_o\neq\mathcal{E}_b$). While, it is required that $\mathcal{G}_b(\mathbf{p})$ is infinitesimally bearing rigid, in the following section, for finite-time estimation of orientation, we will show that $\mathcal{G}_o$ just needs to be connected.

The first problem investigated in this paper is estimating the orientations of the agents in finite time.

Problem 1: For a system of n agents whose orientation sensing graph is characterized by $\mathcal{G}_o$, design an estimation law such that $\hat{\mathbf{Q}}_i\to\mathbf{Q}^c\mathbf{Q}_i$, exponentially converges in finite time to a constant coordinate rotation $\mathbf{Q}^c\in\mathrm{SO}(d)$ by using only relative orientation information (4) $\forall i=1,\ldots,n$.

Remark 1: It may be noted that the objective of the orientation estimator in Problem 1 is to estimate $\mathbf{Q}_i$, up to a common orientation of $\mathbf{Q}^c\ \forall i=1,\ldots,n$.

Once the agents' orientations have been determined, we now formally state the finite-time formation control problem.

Problem 2: Under Assumptions 1 and 2, design a bearing-only control law for each agent i using only bearing vectors $\{\mathbf{g}_{ij}^i\}_{j\in\mathcal{N}_i}$, and the estimated orientations $\hat{\mathbf{Q}}_i$ that achieve the desired formation in finite time.

IV. Finite-Time Orientation Estimation

This section establishes the finite-time convergence of the estimated orientations for undirected and rooted acyclic digraphs. Furthermore, we derive the upper bounds on the settling time of convergence in the two orientation estimation scenarios.

A. Proposed Orientation Estimation Law

For each agent i, we introduce an auxiliary matrix $\mathbf{P}_i\in\mathbb{R}^{d\times d}$ and the estimated orientation $\hat{\mathbf{Q}}_i(t)$ is obtained from $\mathbf{P}_i(t)$ by applying the GSOP such that $\hat{\mathbf{Q}}_i^T=\mathrm{GSOP}(\mathbf{P}_i^T)$. We propose a distributed orientation estimator for each agent as follows:

$$\dot{\mathbf{P}}_i(t)=-\mathbf{P}_i(t)\mathcal{S}^T(\omega_i^i)+\sum_{j\in\mathcal{N}_i}\frac{\mathbf{P}_j(t)\mathbf{Q}_{ij}^T(t)-\mathbf{P}_i(t)}{||\mathbf{P}_j\mathbf{Q}_{ij}^T-\mathbf{P}_i||_F^\alpha}\quad(5)$$

where the scalar $\alpha\in(0,1)$; $\omega_i^i\in\mathbb{R}^d$ denotes the angular velocity of agent i measured in the ith local coordinate frame, ${}^i\Sigma$; $\mathcal{S}(\omega_i^i)$ is the matrix representation of cross product, i.e., $\mathcal{S}(\omega_i^i)\mathbf{v}:=\omega_i^i\times\mathbf{v},\forall\mathbf{v}\in\mathbb{R}^d$. For example, in $\mathbb{R}^3$, $\mathcal{S}(\omega)$ can be given as:

$$\mathcal{S}(\omega)=\begin{bmatrix}0&-\omega_z&\omega_y\\ \omega_z&0&-\omega_x\\ -\omega_y&\omega_x&0\end{bmatrix}$$

[30]. Note that for $0<\alpha<1$, (5) is a continuous estimation law. Furthermore, in (5), agent j sends $\mathbf{P}_j$ to i via communication, for all $j\in\mathcal{N}_i$.

Since the objective of the estimation law (5) is the finite-time estimation of orientation, only the steady-state values of $\mathbf{P}_i,\ i\in\mathcal{V}$, are of interest. Thus, we do not impose any condition on the initial values. Although in the following subsection, we show that the initial values should not belong to a set of Lesbegue measure zero for the estimation of orientation. At any instant of time, one may derive $\hat{\mathbf{Q}}_i(t)$ from $\mathbf{P}_i(t)$ by the GSOP, which contains a finite number of operations, i.e., $O(d^3)$. We assume that $\hat{\mathbf{Q}}_i(t)$ are computed with negligible delay. Note that the Gram–Schmidt process can be finally deployed once the auxiliary variables are in steady state as we will show in the following section.

To analyze the estimation law (5) we present the following lemma.

Lemma 4: The denominator of the right-hand side of (5) can be calculated as follows:

$$\|\mathbf{P}_j\mathbf{Q}_{ij}^T-\mathbf{P}_i\|_F^\alpha=\|\mathbf{P}_i\mathbf{Q}_i^T-\mathbf{P}_j\mathbf{Q}_j^T\|_F^\alpha$$

Proof: Since the trace function is invariant under cyclic permutations, we have

$$\begin{aligned}&\|\mathbf{P}_i\mathbf{Q}_i^T-\mathbf{P}_j\mathbf{Q}_j^T\|_F^\alpha\\&=\mathrm{tr}\{(\mathbf{P}_i\mathbf{Q}_i^T-\mathbf{P}_j\mathbf{Q}_j^T)^T(\mathbf{P}_i\mathbf{Q}_i^T-\mathbf{P}_j\mathbf{Q}_j^T)\}^{\alpha/2}\\&=\mathrm{tr}\{\mathbf{Q}_i\mathbf{Q}_i^T(\mathbf{P}_i\mathbf{Q}_i^T-\mathbf{P}_j\mathbf{Q}_j^T)^T(\mathbf{P}_i\mathbf{Q}_i^T-\mathbf{P}_j\mathbf{Q}_j^T)\}^{\alpha/2}\\&=\mathrm{tr}\{\mathbf{Q}_i^T(\mathbf{P}_i\mathbf{Q}_i^T-\mathbf{P}_j\mathbf{Q}_j^T)^T(\mathbf{P}_i\mathbf{Q}_i^T-\mathbf{P}_j\mathbf{Q}_j^T)\mathbf{Q}_i\}^{\alpha/2}\\&=\mathrm{tr}\{(\mathbf{P}_i-\mathbf{P}_j\mathbf{Q}_j^T\mathbf{Q}_i)^T(\mathbf{P}_i-\mathbf{P}_j\mathbf{Q}_j^T\mathbf{Q}_i)\}^{\alpha/2}\\&=\|\mathbf{P}_j\mathbf{Q}_{ij}^T-\mathbf{P}_i\|_F^\alpha.\end{aligned}$$

■

Hence, (5) can be written as

$$\dot{\mathbf{P}}_i(t)\mathbf{Q}_i^T(t) = -\mathbf{P}_i(t)\mathcal{S}^T(\omega_i^i)\mathbf{Q}_i^T(t) + \sum_{j\in\mathcal{N}_i} \frac{\mathbf{P}_j(t)\mathbf{Q}_j^T(t) - \mathbf{P}_i(t)\mathbf{Q}_i^T(t)}{||\mathbf{P}_j(t)\mathbf{Q}_j^T(t) - \mathbf{P}_i(t)\mathbf{Q}_i^T(t)||_F^\alpha}.$$

Consider the rotation transformation $\mathbf{P}_i(t) = \mathbf{S}_i(t)\mathbf{Q}_i(t)$ and note that

$$\dot{\mathbf{S}}_i = d/dt(\mathbf{P}_i\mathbf{Q}_i^T) = \dot{\mathbf{P}}_i\mathbf{Q}_i^T + \mathbf{P}_i\mathcal{S}^T(\omega_i^i)\mathbf{Q}_i^T$$

where we use the relation $\dot{\mathbf{Q}}_i^T = \mathcal{S}^T(\omega_i^i)\mathbf{Q}_i^T$ [30]. Thus, the above equation can be rewritten as

$$\dot{\mathbf{S}}_i(t) = \sum_{j\in\mathcal{N}_i} (\mathbf{S}_j - \mathbf{S}_i)/\|\mathbf{S}_i - \mathbf{S}_j\|_F^\alpha. \tag{6}$$

B. Undirected Graph

Let $\mathbf{S} = [\mathbf{S}_1^T, \ldots, \mathbf{S}_n^T]^T \in \mathbb{R}^{dn\times d}$ be the stacked matrix of the transformed matrix variables. Thus, (6) can be rewritten as

$$\dot{\mathbf{S}}(t) = -(\bar{\mathcal{L}}_o \otimes \mathbf{I}_d)\mathbf{S}(t) \tag{7}$$

where the matrix $\bar{\mathcal{L}}_o = [\bar{l}_{ij}]$ is defined as

$$\bar{l}_{ij} = \begin{cases} 0, \; (i,j)\in\mathcal{E}_o, \; i\neq j, \; \mathbf{S}_i = \mathbf{S}_j \text{ or } (i,j)\notin\mathcal{E}_o, \; i\neq j \\ -1/\|\mathbf{S}_i - \mathbf{S}_j\|_F^\alpha, \quad (i,j)\in\mathcal{E}_o, \; i\neq j, \; \mathbf{S}_i\neq\mathbf{S}_j \\ \sum_{k\in\mathcal{N}_i}\bar{l}_{ik}, \quad i=j, i\in\mathcal{V}\end{cases}$$

which is a weighted Laplacian for the graph $\mathcal{G}_o$.

Since $(\mathbf{1}_n\otimes\mathbf{I}_d)^T\dot{\mathbf{S}}(t) = -(\mathbf{1}_n\otimes\mathbf{I}_d)^T(\bar{\mathcal{L}}_o\otimes\mathbf{I}_d)\mathbf{S}(t) = \mathbf{0}$, $(\mathbf{1}_n\otimes\mathbf{I}_d)^T\mathbf{S}(t)$ is invariant under (7). Let $\mathbf{S}^c = (1/n)(\mathbf{1}_n \otimes\mathbf{I}_d)^T\mathbf{S}(t)\in\mathbb{R}^{d\times d}$, $\mathbf{S}_i(t) = \mathbf{S}^c + \boldsymbol{\delta}_i(t)$, and let $\boldsymbol{\delta}(t) = [\boldsymbol{\delta}_1^T,\ldots,\boldsymbol{\delta}_n^T]^T\in\mathbb{R}^{dn\times d}$. Since $\mathbf{S}^c$ is time invariant, it follows that $\dot{\boldsymbol{\delta}}_i(t) = \dot{\mathbf{S}}_i(t)$. Note that $\boldsymbol{\delta}_i - \boldsymbol{\delta}_j = \mathbf{S}_i - \mathbf{S}_j$.

Theorem 1: Under the estimation law (5) and assuming that $\mathcal{G}_o$ is a connected undirected graph, $\mathbf{S}(t)$ globally asymptotically converges to $(\mathbf{1}_n\otimes\mathbf{I}_d)\text{Ave}\{\mathbf{S}_i(0)\}_{i\in\mathcal{V}}$ in finite time.

Proof: Consider a Lyapunov candidate function

$$V(t) = (1/2)\sum_{i=1}^n \|\boldsymbol{\delta}_i\|_F^2 = (1/2)\sum_{i=1}^n \text{tr}\big(\boldsymbol{\delta}_i^T\boldsymbol{\delta}_i\big). \tag{8}$$

Note that V is radially unbounded, positive definite, continuously differentiable, and $V=0$ in $\mathcal{S}_o := \big\{\{\mathbf{S}_i\}_{i\in\mathcal{V}} \,|\, \mathbf{S}_i = \mathbf{S}^c, \forall i\in\mathcal{V}\big\}$. The time derivative of V is given as

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^n \text{tr}\big(\boldsymbol{\delta}_i^T\dot{\boldsymbol{\delta}}_i\big) = \text{tr}\left\{\sum_{i=1}^n \boldsymbol{\delta}_i^T\dot{\boldsymbol{\delta}}_i\right\} \\ &= -\text{tr}\left\{\sum_{i=1}^n \boldsymbol{\delta}_i^T \sum_{j\in\mathcal{N}_i}(\boldsymbol{\delta}_i - \boldsymbol{\delta}_j)/\|\boldsymbol{\delta}_i - \boldsymbol{\delta}_j\|_F^\alpha\right\} \\ &= -\text{tr}\left\{\sum_{(i,j)\in\mathcal{E}_o}\big(\boldsymbol{\delta}_i^T - \boldsymbol{\delta}_j^T\big)(\boldsymbol{\delta}_i - \boldsymbol{\delta}_j)/\|\boldsymbol{\delta}_i - \boldsymbol{\delta}_j\|_F^\alpha\right\} \\ &= -\sum_{(i,j)\in\mathcal{E}_o}\|\boldsymbol{\delta}_i - \boldsymbol{\delta}_j\|_F^2/\|\boldsymbol{\delta}_i - \boldsymbol{\delta}_j\|_F^\alpha \\ &= -\sum_{(i,j)\in\mathcal{E}_o}\big(\|\boldsymbol{\delta}_i - \boldsymbol{\delta}_j\|_F^2\big)^{(2-\alpha)/2} \\ &\leq -\left(\sum_{(i,j)\in\mathcal{E}_o}\|\boldsymbol{\delta}_i - \boldsymbol{\delta}_j\|_F^2\right)^{(2-\alpha)/2} \end{aligned} \tag{9}$$

$$\begin{aligned} &\leq -\left[\text{tr}\left\{\sum_{(i,j)\in\mathcal{E}_o}(\boldsymbol{\delta}_i^T - \boldsymbol{\delta}_j^T)(\boldsymbol{\delta}_i - \boldsymbol{\delta}_j)\right\}\right]^{(2-\alpha)/2} \\ &\leq -\Big[\text{tr}\big\{\boldsymbol{\delta}^T(\mathcal{L}_o\otimes\mathbf{I}_d)\boldsymbol{\delta}\big\}\Big]^{(2-\alpha)/2} \\ &\leq -\left[\sum_{k=1}^d[\boldsymbol{\delta}]_k^T(\mathcal{L}_o\otimes\mathbf{I}_d)[\boldsymbol{\delta}]_k\right]^{(2-\alpha)/2}\end{aligned} \tag{10}$$

where $[\boldsymbol{\delta}]_k$ denotes the kth$(1\leq k\leq d)$ column vector of $\boldsymbol{\delta}$, and the inequality (9) is derived by applying Lemma 1 with $\frac{1}{2} < \frac{2-\alpha}{2} < 1$. Under the assumption that $\mathcal{G}_o$ is connected undirected, $\mathcal{L}_o\otimes\mathbf{I}_d$ has d zero eigenvalues and $\mathcal{N}(\mathcal{L}_o\otimes\mathbf{I}_d) = \text{span}\{\text{Range}(\mathbf{1}_n\otimes\mathbf{I}_d)\}$. Since $[\boldsymbol{\delta}]_k \perp \text{Range}(\mathbf{1}_n\otimes\mathbf{I}_d)$ $\forall k = 1,\ldots,d$, one has

$$[\boldsymbol{\delta}]_k^T(\mathcal{L}_o\otimes\mathbf{I}_d)[\boldsymbol{\delta}]_k \geq \lambda_{d+1}[\boldsymbol{\delta}]_k^T[\boldsymbol{\delta}]_k \quad \forall k=1,\ldots,d$$

where λ_{d+1} is the smallest nonzero eigenvalue of $\mathcal{L}_o\otimes\mathbf{I}_d$. Substituting the above inequalities into (10) yields the following:

$$\begin{aligned}\dot{V}(t) &\leq -\left[\lambda_{d+1}\sum_{k=1}^d\|[\boldsymbol{\delta}]_k\|^2\right]^{(2-\alpha)/2} \\ &\leq -\lambda_{d+1}^{(2-\alpha)/2}\left[\sum_{k=1}^d\sum_{i=1}^n\|[\boldsymbol{\delta}_i]_k\|^2\right]^{(2-\alpha)/2} \\ &\leq -\lambda_{d+1}^{(2-\alpha)/2}\left[\sum_{i=1}^n\text{tr}\big(\boldsymbol{\delta}_i^T\boldsymbol{\delta}_i\big)\right]^{(2-\alpha)/2} \\ &\leq -\lambda_{d+1}^{(2-\alpha)/2}\big(2V(t)\big)^{(2-\alpha)/2} \leq -\kappa V(t)^{(2-\alpha)/2}\end{aligned} \tag{11}$$

where $\kappa = (2\lambda_{d+1})^{(2-\alpha)/2}$. It follows from Lemma 2 and (11) that $V(t)$ converges to 0 in finite time. In other words, $\{\mathbf{S}_i\}_{i\in\mathcal{V}}$ converges to the invariant set $\mathcal{S}_o$. As a result, it follows that $\mathbf{S}_i(t)\,\forall i\in\mathcal{V}$, globally converges to $\mathbf{S}^c = \text{Ave}\{\mathbf{S}_i(0)\}_{i\in\mathcal{V}}$ with settling time $T_o \leq V(0)^{1-\frac{2-\alpha}{2}}/(\kappa(1-\frac{2-\alpha}{2})) = 2V(0)^{\frac{\alpha}{2}}/(\kappa\alpha)$. ∎

Since $\hat{\mathbf{Q}}_i$ is derived from the auxiliary variable $\mathbf{P}_i$ by the Gram–Schmidt orthognormalization process, i.e., $\hat{\mathbf{Q}}_i^T = \text{GSOP}(\mathbf{P}_i^T)$, we have the following corollary.

Corollary 1: Suppose $\mathcal{G}_o$ is connected, under the orientation estimation law (5), there exists a constant $\mathbf{Q}^c\in\text{SO}(d)$ such that $\hat{\mathbf{Q}}_i$ globally converges to $\mathbf{Q}^c\mathbf{Q}_i$ in finite time, for almost all initial value $\mathbf{S}(0)\in\mathbb{R}^{dn\times d}$, for all $i = 1,\ldots,n$.

Proof: The rotation matrices $\mathbf{Q}_{\mathbf{S}_i}^T(t), \hat{\mathbf{Q}}_i^T(t)\in\mathbb{R}^{d\times d}$ are, respectively, derived from $\mathbf{S}_i^T(t)$ and $\mathbf{P}_i^T(t)$ by the GSOP. Note that the orthonormal property of a set of orthonormal vectors is preserved under coordination rotations, it can be verified that $\hat{\mathbf{Q}}_i(t) = \mathbf{Q}_{\mathbf{S}_i}(t)\mathbf{Q}_i$. From Theorem 1, $\mathbf{S}_i(t)$ globally exponentially converges to $\mathbf{S}^c \in \mathbb{R}^{d\times d}$ and, hence, $\hat{\mathbf{Q}}_i(t)$ reaches $\mathbf{Q}^c\mathbf{Q}_i\ \forall i = 1,\ldots,n$, where $\mathbf{Q}^c \in \mathrm{SO}(d)$ is derived from $\mathbf{S}^c$ by the GSOP. In other words, the actual orientations of the agents are determined in a finite time up to a constant coordinate rotation $\mathbf{Q}^c$.

For the validity of estimated orientation, i.e., $\mathbf{Q}^c \in \mathrm{SO}(d)$, convergence to the zero vector of any column vectors of the steady state $\mathbf{S}^c = \mathrm{Ave}\{\mathbf{S}_i(0)\}$, or linear dependence of the column vectors of $\mathbf{S}^c$ are undesired. Furthermore, the set of column vectors of initial matrix auxiliary variables $\mathbf{S}_i(0) \in \mathbb{R}^{d\times d}$, $i = 1,\ldots,n$, leading to nonexistence of solution is a set of Lesbegue measure zero in $\mathbb{R}^{dn}$ [19]. This completes the proof. ■

From Corollary 1, the estimation law (5) *almost globally asymptotically finite-time* solves the Problem 1.

C. Directed Graph

For directed orientation sensing graphs, we impose the following assumption.

Assumption 3: The orientation sensing graph $\mathcal{G}_o$ is a rooted acyclic digraph.

Theorem 2: Under the Assumption 3, $\mathbf{S}_i(t)$ of the dynamics (6) globally exponentially converges to $\mathbf{S}_1(0)$ in finite time, for all $i = 2,\ldots,n$.

Proof: Since the root node (say node 1) has no neighbors, its estimated state, $\mathbf{S}_1$, is unchanged under the estimation law (6), i.e., $\mathbf{S}_1(t) = \mathbf{S}_1(0) = \mathbf{P}_1(0)\mathbf{Q}_1^T(0)$. We prove Theorem 2 by using mathematical induction as follows.

i) For Agent 2 having Agent 1 as its neighbor

$$\dot{\mathbf{S}}_2(t) = (\mathbf{S}_1 - \mathbf{S}_2)/\|\mathbf{S}_2 - \mathbf{S}_1\|_F^\alpha. \tag{12}$$

Consider a $\mathcal{C}^1$ Lyapunov candidate function

$$V_2(t) = (1/2)\|\mathbf{S}_2 - \mathbf{S}_1\|_F^2 = (1/2)\mathrm{tr}\left((\mathbf{S}_2 - \mathbf{S}_1)^T(\mathbf{S}_2 - \mathbf{S}_1)\right)$$

which is positive definite and radially unbounded. The derivative is

$$\begin{aligned}\dot{V}_2(t) &= \mathrm{tr}\big((\mathbf{S}_2 - \mathbf{S}_1)^T(\dot{\mathbf{S}}_2 - \dot{\mathbf{S}}_1)\big)\\ &= -\mathrm{tr}\big((\mathbf{S}_1 - \mathbf{S}_2)^T(\mathbf{S}_1 - \mathbf{S}_2)\big)/\|\mathbf{S}_2 - \mathbf{S}_1\|_F^\alpha\\ &= -(2/2^{\alpha/2})V_2(t)^{-\alpha/2}V_2(t) = -\gamma_2 V_2(t)^{(2-\alpha)/2}\end{aligned}$$

where $\gamma_2 = 2^{(2-\alpha)/2}$. Thus, there exists $T_2 \leq V_2(0)^{1-\frac{2-\alpha}{2}}/(\gamma_2(1-\frac{2-\alpha}{2})) = 2V_2(0)^{\frac{\alpha}{2}}/(\gamma_2\alpha)$ such that $\mathbf{S}_2(t) \to \mathbf{S}_1(0)$ as $t \to T_2$.

ii) Now, we assume that it is true for $k-1$ agents ($3 \leq k \leq n$), i.e., $\mathbf{S}_i = \mathbf{S}_1(0)$ for $t \geq T_i, \forall i = 1,\ldots,k-1$. Consider agent k, which has one or some neighbors in $\{1,\ldots,k-1\}$; we will show that there exists a finite time $T_k \geq T_i' \triangleq \max_{i\in\mathcal{N}_k}(T_i)$ such that $\mathbf{S}_k = \mathbf{S}_1$ for $t \geq T_k$. Consider the estimation dynamics of agent k

$$\dot{\mathbf{S}}_k(t) = -\sum_{i\in\mathcal{N}_k}(\mathbf{S}_k - \mathbf{S}_i)/\|\mathbf{S}_k - \mathbf{S}_i\|_F^\alpha. \tag{13}$$

Consider a Lyapunov function

$$\begin{aligned}V_k(t) &= (1/2)\sum_{i\in\mathcal{N}_k}\|\mathbf{S}_k - \mathbf{S}_i\|_F^2\\ &= (1/2)\sum_{i\in\mathcal{N}_k}\mathrm{tr}\big((\mathbf{S}_k - \mathbf{S}_i)^T(\mathbf{S}_k - \mathbf{S}_i)\big)\end{aligned} \tag{14}$$

which is also positive definite, radially unbounded, and continuously differentiable. The derivative of V_k is given as

$$\dot{V}_k(t) = \sum_{i\in\mathcal{N}_k}\mathrm{tr}\Big((\mathbf{S}_k - \mathbf{S}_i)^T(\dot{\mathbf{S}}_k - \dot{\mathbf{S}}_i)\Big). \tag{15}$$

Since the states of a continuous system are bounded in finite time and V_k is continuous, thus, it is bounded for $t \leq T_i'$. For $t > T_i'$, $\dot{\mathbf{S}}_i = \mathbf{0}$ and $\mathbf{S}_i = \mathbf{S}_1, \forall i \in \mathcal{N}_k$. Then, it follows from (15) that

$$\begin{aligned}\dot{V}_k(t) &= |\mathcal{N}_k|\mathrm{tr}\big((\mathbf{S}_k - \mathbf{S}_1)^T\dot{\mathbf{S}}_k\big)\\ &= -|\mathcal{N}_k|\mathrm{tr}\left((\mathbf{S}_k - \mathbf{S}_1)^T\sum_{i\in\mathcal{N}_k}\frac{(\mathbf{S}_k - \mathbf{S}_i)}{\|\mathbf{S}_k - \mathbf{S}_i\|_F^\alpha}\right)\\ &= -|\mathcal{N}_k|\sum_{i\in\mathcal{N}_k}\mathrm{tr}\left((\mathbf{S}_k - \mathbf{S}_i)^T\frac{(\mathbf{S}_k - \mathbf{S}_i)}{\|\mathbf{S}_k - \mathbf{S}_i\|_F^\alpha}\right)\\ &= -(2|\mathcal{N}_k|/2^{\alpha/2})V_k(t)^{-\alpha/2}V_k(t)\\ &= -\gamma_k V_k(t)^{(2-\alpha)/2}\end{aligned}$$

where $\gamma_k = 2|\mathcal{N}_k|/2^{\alpha/2}$. Thus, there exists $T_k \leq T_i' + 2V_k(T_i')^{\alpha/2}/(\gamma_k\alpha)$ such that $\mathbf{S}_k(T_k) = \mathbf{S}_1(0)$ for $t \geq T_k$.

iii) By following the steps until $k = n$ we show that $\mathbf{S}_k(t)$ globally exponentially converge to $\mathbf{S}_1(0)$ in finite time, for all $k = 2,\ldots,n$. ■

Corollary 2: Under Assumption 3 and the orientation estimation law (5), there exists a constant $\mathbf{Q}^c \in \mathrm{SO}(d)$ such that $\hat{\mathbf{Q}}_i$ globally exponentially converges to $\mathbf{Q}^c\mathbf{Q}_i$ in finite time if and only if the matrix $\mathbf{P}_1(0)$ is nonsingular, for all $i = 1,\ldots,n$.

Proof: By Theorem 2, $\mathbf{S}_i$ converges to $\mathbf{S}_1 \in \mathbb{R}^{d\times d}$ and, hence, $\mathbf{P}_i$ converges to $\mathbf{S}_1(0)\mathbf{Q}_i\ \forall i = 1,\ldots,n$. By using the similar arguments on the invariant property of the GSOP under coordinate rotations, if $\mathbf{P}_1(0)$ is nonsingular and $\mathbf{Q}_{\mathbf{P}_1}^T$ is derived from $\mathbf{P}_1^T(0)$ by GSOP [$\mathbf{Q}_{\mathbf{P}_1}^T = \mathrm{GSOP}(\mathbf{P}_1^T(0))$], $\hat{\mathbf{Q}}_i \forall i \in \{2,\ldots,n\}$ globally converge to $\mathbf{Q}_{\mathbf{P}_1}\mathbf{Q}_1^T(0)\mathbf{Q}_i$ in a finite time T_d. Consequently, the agents' orientations are globally exponentially finite-time determined up to a constant coordinate rotation $\mathbf{Q}^c = \mathbf{Q}_{\mathbf{P}_1}\mathbf{Q}_1^T(0)$. We have, thus, established the sufficiency of the full-rank condition of the matrix auxiliary variable $\mathbf{P}_1$.

If $\mathbf{P}_1(0)$ is singular, then $\mathbf{P}_1(0)\mathbf{Q}_1^T(0)\mathbf{Q}_i$ is singular. As a result, there does not exist $\hat{\mathbf{Q}} \in \mathrm{SO}(d)$ such that $\hat{\mathbf{Q}}$ is derived from nonsingular matrix $\mathbf{P}_1(0)\mathbf{Q}_1^T(0)\mathbf{Q}_i$ by the GSOP. Thus, the necessity follows directly. ■

Algorithm 1: Finite-Time Bearing-Only Formation Control via Orientation Estimation.

```
Initialization: t ← 0, P_i(0) ∈ ℝ^{d×d}, p_i(0) ∈ ℝ^d
Estimation loop:
 repeat
   for all i ∈ V do
     P_i(t) ← integrate (5)
Start control loop (at t_c ≥ T_o, convergence time of
orientation estimation):
      if t ≥ T_0
        Q̂_i^T ← GSOP(P_i^T)
        p_i(t) ← integrate (16)
      end if
   end for
 until t ≥ T_c (T_c convergence time of formation
 control).
End control loop.
End estimation loop.
```

V. Finite-Time Bearing-Only Formation Control

This section establishes almost global finite-time convergence of the desired formation. In doing this, we show that it is sufficient for the agents to achieve the desired formation shape in a finite time without a common sense of the global coordinate frame by finite-time estimation of their orientations.

A. Proposed Control Law

For each agent i, we propose the following control law:

$$\mathbf{u}_i^i = -\sum_{j\in\mathcal{N}_j} \mathbf{P}_{\mathbf{g}_{ij}^i}\hat{\mathbf{Q}}_i^T \operatorname{sig}\big(\hat{\mathbf{Q}}_i \mathbf{P}_{\mathbf{g}_{ij}^i}\hat{\mathbf{Q}}_i^T \mathbf{g}_{ij}^*\big)^\alpha \tag{16}$$

where $\alpha \in (0,1)$ is a positive constant, and $\hat{\mathbf{Q}}_i$ is the estimated orientation of agent i obtained by a finite-time orientation estimator. Since $0 < \alpha < 1$ the control law is continuous. In (16), it is straightforward to show that the local projection matrix can be rewritten as $\mathbf{P}_{\mathbf{g}_{ij}^i} = \mathbf{I}_d - \mathbf{g}_{ij}^i(\mathbf{g}_{ij}^i)^T = \mathbf{Q}_i^T\mathbf{P}_{\mathbf{g}_{ij}}\mathbf{Q}_i$, where $\mathbf{g}_{ij}^i$ represents coordinates of $\mathbf{g}_{ij}$ expressed in the ith coordinate frame.

Remark 2: In the finite-time formation control via orientation estimation scheme, the finite-time orientation estimation (5) and the finite-time formation control (16) run in a sequential way. In particular, once the auxiliary matrices $\mathbf{P}_i$ are in steady state, the GSOP is finally deployed to construct the estimation orientations $\hat{\mathbf{Q}}_i$. Then, the formation control (16) is activated by using the estimated orientations $\hat{\mathbf{Q}}_i$. The formation control strategy via orientation estimation is illustrated in Algorithm 1.

Assume that the control loop is activated at $t = t_c \geq T_o$, where T_o is convergence time of the orientation estimation, i.e., $\hat{\mathbf{Q}}_i = \mathbf{Q}^c\mathbf{Q}_i$, $\forall i \in \mathcal{V}$, for $t \geq T_o$.

The dynamics of each agent i expressed in the global coordinate frame are given by

$$\dot{\mathbf{p}}_i = \mathbf{Q}_i\mathbf{u}_i^i \quad \forall i \in \mathcal{V}. \tag{17}$$

Substituting $\mathbf{u}_i^i$ from (16) and $\hat{\mathbf{Q}}_i = \mathbf{Q}^c\mathbf{Q}_i$ into (17) yields the following:

$$\begin{aligned}\dot{\mathbf{p}}_i &= -\mathbf{Q}_i\sum_{j\in\mathcal{N}_j}\mathbf{Q}_i^T\mathbf{P}_{\mathbf{g}_{ij}}\mathbf{Q}_i\hat{\mathbf{Q}}_i^T\operatorname{sig}\big(\hat{\mathbf{Q}}_i\mathbf{Q}_i^T\mathbf{P}_{\mathbf{g}_{ij}}\mathbf{Q}_i\hat{\mathbf{Q}}_i^T\mathbf{g}_{ij}^*\big)^\alpha \\ &= -\sum_{j\in\mathcal{N}_j}\mathbf{P}_{\mathbf{g}_{ij}}\mathbf{Q}^{cT}\operatorname{sig}\big(\mathbf{Q}^c\mathbf{P}_{\mathbf{g}_{ij}}\mathbf{Q}^{cT}\mathbf{g}_{ij}^*\big)^\alpha \\ &= -\sum_{j\in\mathcal{N}_j}\mathbf{P}_{\mathbf{g}_{ij}}(\mathbf{Q}^c)^T\operatorname{sig}\big(\mathbf{Q}^c\mathbf{P}_{\mathbf{g}_{ij}}\mathbf{g}_{ij}^c\big)^\alpha\end{aligned} \tag{18}$$

where $\mathbf{g}_{ij}^c = (\mathbf{Q}^c)^T\mathbf{g}_{ij}^*$.

Let $\mathbf{g}^c = [\mathbf{g}_1^c, \ldots, \mathbf{g}_m^c]^T \in \mathbb{R}^{dm}$. We can write the n-agent system under control law (16) in the compact form as follows:

$$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}) = \bar{\mathbf{H}}^T\operatorname{diag}(\mathbf{P}_{\mathbf{g}_k}\mathbf{Q}^{cT})\operatorname{sig}\big(\operatorname{diag}(\mathbf{Q}^c\mathbf{P}_{\mathbf{g}_k})\mathbf{g}^c\big)^\alpha. \tag{19}$$

We follow techniques similar to those in [7] and [11] to investigate the equilibrium set and the stability analyses of the dynamics (19) as follows.

B. Equilibrium Set

We denote $\bar{\mathbf{p}} = (1/n)\sum_{i=1}^n \mathbf{p}_i = (1/n)(\mathbf{1}_n \otimes \mathbf{I}_d)^T\mathbf{p}$ as the centroid and $s = \sqrt{(1/n)\sum_{i=1}^n \|\mathbf{p}_i - \bar{\mathbf{p}}\|^2} = (1/\sqrt{n})\|\mathbf{p} - \mathbf{1}_n \otimes \bar{\mathbf{p}}\|$ as the scale of the scale of the formation, respectively.

Lemma 5: The centroid and scale of the formation are invariant under control law (19).

Proof: We rewrite the system dynamics (19) as

$$\dot{\mathbf{p}} = \tilde{\mathbf{R}}^T\operatorname{diag}(\mathbf{Q}^{cT})\operatorname{sig}\Big(\operatorname{diag}(\mathbf{Q}^c\mathbf{P}_{\mathbf{g}_k})\mathbf{g}^c\Big)^\alpha \tag{20}$$

where $\tilde{\mathbf{R}} = \operatorname{diag}(\mathbf{P}_{\mathbf{g}_k})\bar{\mathbf{H}}$. From (2), we have $(\operatorname{diag}(\|\mathbf{z}_k\|) \otimes \mathbf{I}_d)\tilde{\mathbf{R}} = \mathbf{R}$. Consequently, it follows that $\mathcal{N}(\tilde{\mathbf{R}}) = \mathcal{N}(\mathbf{R})$. As a result, from Lemma 3, we obtain $\dot{\mathbf{p}} \perp \operatorname{span}\{\operatorname{Range}(\mathbf{1}_n \otimes \mathbf{I}_d), \mathbf{p}\}$. Thus, the formation's centroid and scale are invariant by the following equalities:

$$\dot{\bar{\mathbf{p}}} = (1/n)(\mathbf{1}_n \otimes \mathbf{I}_d)^T\dot{\mathbf{p}} = \mathbf{0}$$

$$\dot{s} = \frac{1}{\sqrt{n}}\frac{(\mathbf{p} - \mathbf{1}_n \otimes \bar{\mathbf{p}})^T}{\|\mathbf{p} - \mathbf{1}_n \otimes \bar{\mathbf{p}}\|}\dot{\mathbf{p}} = 0.$$

■

The equilibrium set of the system (19) is given in the following lemma.

Lemma 6: The system (19) has two isolated equilibria, $\mathbf{p}^c$ corresponding to $\mathbf{g}_k = \mathbf{g}_k^c\ \forall k = 1,\ldots,m$, and $\mathbf{p}\prime$ corresponding to $\mathbf{g}_k = -\mathbf{g}_k^c\ \forall k = 1,\ldots,m$.

Proof: Let $\mathcal{S}_e = \{\mathbf{p} \in \mathbb{R}^{dn} \,|\, \dot{\mathbf{p}} = 0\}$. The equilibria of (19) can be found by assigning $\dot{\mathbf{p}} = \mathbf{0}$. This leads to $\operatorname{diag}(\mathbf{Q}^c\mathbf{P}_{\mathbf{g}_k})\mathbf{g}^c = \mathbf{0}$, which is equivalent to $\mathbf{P}_{\mathbf{g}_k}\mathbf{g}_k^c = \mathbf{0}\ \forall k = 1,\ldots,m$.

The remaining proof is followed from a similar argument as in [7, Th. 10] by the facts that the desired framework is infinitesimally bearing rigid, and the centroid and scale of the formation are invariant under (19). ■

Let $\boldsymbol{\delta}_i = \mathbf{p}_i - \mathbf{p}_i^c$, and $\boldsymbol{\delta} = [\boldsymbol{\delta}_1^T, \ldots, \boldsymbol{\delta}_n^T]^T$. Thus, $\dot{\boldsymbol{\delta}}_i = \dot{\mathbf{p}}_i$; (19) can be rewritten as

$$\dot{\boldsymbol{\delta}} = \mathbf{f}(\boldsymbol{\delta}) = \bar{\mathbf{H}}^T \mathrm{diag}(\mathbf{P}_{\mathbf{g}_k} \mathbf{Q}^{cT}) \mathrm{sig}\Big(\mathrm{diag}(\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k}) \mathbf{g}^c\Big)^{\alpha}. \quad (21)$$

Let $\mathbf{r}(t) = \mathbf{p}(t) - \mathbf{1}_n \otimes \bar{\mathbf{p}}(t)$, and the steady state $\mathbf{r}^c = \mathbf{p}^c - \mathbf{1}_n \otimes \bar{\mathbf{p}}^c$. Since the centroid of the formation is invariant (see Lemma 5), it can be shown that the solution trajectory of (21) lies on the surface of the sphere $\mathcal{S} = \{\boldsymbol{\delta} \in \mathbb{R}^{dn} \,|\, \|\boldsymbol{\delta} + \mathbf{r}^c\| = \|\mathbf{r}^c\|\}$ [11].

Note that the system (21) has two equilibrium points $\boldsymbol{\delta} = \mathbf{0}$ and $\boldsymbol{\delta} = -2\mathbf{r}^c$ [7, Th. 10]. We investigate the stability of these equilibria in the following section.

C. Stability Analysis

In this part, we establish the almost global convergence of the desired formation under the control law (16). In particular, the agents asymptotically achieved the target formation if the initial configuration is not in a set of Lebesgue measure zero in $\mathbb{R}^{dn}$.

Theorem 3: Under the Assumption 2 and the control law (16), the desired equilibrium $\mathbf{p} = \mathbf{p}^c$ of system (19) is asymptotically stable.

Proof: Consider the $\mathcal{C}^1$ Lyapunov candidate function $V = \frac{1}{2}\|\mathbf{p} - \mathbf{p}^c\|^2$, which is positive definite and radially unbounded. The time derivative of V along the trajectory of (21) is

$$\begin{aligned}
\dot{V}(\mathbf{p}) &= (\mathbf{p} - \mathbf{p}^c)^T \bar{\mathbf{H}}^T \mathrm{diag}\big(\mathbf{P}_{\mathbf{g}_k} (\mathbf{Q}^c)^T\big) \mathrm{sig}\big(\mathrm{diag}(\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k}) \mathbf{g}^c\big)^{\alpha} \\
&= -(\mathbf{p}^c)^T \bar{\mathbf{H}}^T \mathrm{diag}\big(\mathbf{P}_{\mathbf{g}_k} (\mathbf{Q}^c)^T\big) \mathrm{sig}\big(\mathrm{diag}(\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k}) \mathbf{g}^c\big)^{\alpha} \\
&= -(\mathbf{z}^c)^T \mathrm{diag}\big(\mathbf{P}_{\mathbf{g}_k} (\mathbf{Q}^c)^T\big) \mathrm{sig}\big(\mathrm{diag}(\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k}) \mathbf{g}^c\big)^{\alpha} \\
&= -\sum_{k=1}^{m} (\mathbf{z}_k^c)^T \mathbf{P}_{\mathbf{g}_k} (\mathbf{Q}^c)^T \mathrm{sig}\,(\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c)^{\alpha} \\
&= -\sum_{k=1}^{m} \|\mathbf{z}_k^c\| (\mathbf{g}_k^{cT} \mathbf{P}_{\mathbf{g}_k} \mathbf{Q}^{cT}) \mathrm{sig}\,(\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c)^{\alpha} \\
&= -\sum_{k=1}^{m} \|\mathbf{z}_k^c\| \sum_{i=1}^{d} \Big| [\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c]_i \Big|^{\alpha+1}. \quad (22)
\end{aligned}$$

Thus, for $\alpha \in (0,1)$, $\dot{V}$ is negative semidefinite, and $\dot{V} = 0$ if and only if $\mathbf{p} = \mathbf{p}^c$ or $\mathbf{p} = \mathbf{p}'$. Based on LaSalle invariant principle, any solution of (19) asymptotically converges to either $\mathbf{p}^c$ or $\mathbf{p}'$. Consider any neighborhood of $\mathbf{p} = \mathbf{p}^c$, which does not contain $\mathbf{p}\prime$, thus, $\dot{V} < 0$ for $\mathbf{p} \neq \mathbf{p}'$ in the region. As a result, $\mathbf{p} = \mathbf{p}^c$ is (locally) asymptotically stable. ■

Lemma 7: The undesired equilibrium $\mathbf{p} = \mathbf{p}'$ of (16) corresponding to $\mathbf{g} = -\mathbf{g}^c$ is unstable.

Proof: Consider the Lyapunov candidate function $V = \frac{1}{2}\|\mathbf{p} - \mathbf{p}'\|^2$, which is positive definite, radially bounded, and continuous differentiable. Since $\bar{\mathbf{H}}\mathbf{p}' = -\mathbf{z}^c$, we have

$$\begin{aligned}
\dot{V}(\mathbf{p}) &= (\mathbf{p} - \mathbf{p}')^T \bar{\mathbf{H}}^T \mathrm{diag}\Big(\mathbf{P}_{\mathbf{g}_k} (\mathbf{Q}^c)^T\Big) \mathrm{sig}\Big(\mathrm{diag}(\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k}) \mathbf{g}^c\Big)^{\alpha} \\
&= -(\mathbf{p}')^T \bar{\mathbf{H}}^T \mathrm{diag}\Big(\mathbf{P}_{\mathbf{g}_k} (\mathbf{Q}^c)^T\Big) \mathrm{sig}\Big(\mathrm{diag}(\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k}) \mathbf{g}^c\Big)^{\alpha} \\
&= (\mathbf{z}^c)^T \mathrm{diag}\Big(\mathbf{P}_{\mathbf{g}_k} (\mathbf{Q}^c)^T\Big) \mathrm{sig}\Big(\mathrm{diag}(\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k}) \mathbf{g}^c\Big)^{\alpha} \\
&= \sum_{k=1}^{m} (\mathbf{z}_k^c)^T \mathbf{P}_{\mathbf{g}_k} (\mathbf{Q}^c)^T \mathrm{sig}\,(\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c)^{\alpha} \\
&= \sum_{k=1}^{m} \|\mathbf{z}_k^c\| (\mathbf{g}_k^{cT} \mathbf{P}_{\mathbf{g}_k} \mathbf{Q}^{cT}) \mathrm{sig}\,(\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c)^{\alpha} \\
&= \sum_{k=1}^{m} \|\mathbf{z}_k^c\| \sum_{i=1}^{d} \Big| [\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c]_i \Big|^{\alpha+1}. \quad (23)
\end{aligned}$$

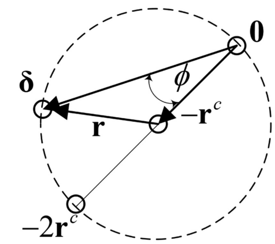


Fig. 1. Interpretation of the proof of Theorem 5: $\boldsymbol{\delta}$ and ϕ.

Thus, $\dot{V} > 0$ in a neighbor of $\mathbf{p}'$. The undesired equilibrium $\mathbf{p} = \mathbf{p}'$ is unstable in recall of the Chetaev instability theorem [31]. ■

Theorem 4: Under the control law (16), the desired equilibrium $\mathbf{p}^c$ of (19) is almost globally asymptotically stable.

Proof: Noting that in the proof of Theorem 3, $\dot{V} < 0$ everywhere in $\mathbb{R}^{dn}$ except at $\mathbf{p}'$, which is a set of measure zero in $\mathbb{R}^{dn}$. This along with Theorem 3 and Lemma 7 completes the proof. ■

D. Finite-Time Convergence Analysis

In this section, we show that under the proposed control scheme, the desired formation is achieved in finite time. For this, the following useful lemma is needed.

Lemma 8: Under Assumption 2 and control law (19), the following inequality holds:

$$\|\mathbf{z}_k\| \leq 2s\sqrt{n-1}\,\forall k = 1, \ldots, n$$

where s is the formation scale.

Proof: One may prove this lemma by following the proof in [7, Corollary 2]. Thus, the proof is omitted. ■

Theorem 5: Under control law (16), $\mathbf{p}$ converges to the desired formation $\mathbf{p}^c$ in a finite time if the initial formation satisfies $\mathbf{p}(t_c) \neq \mathbf{p}'$.

Proof: Define $\epsilon = \min_{k=1,\ldots,m} \|\mathbf{z}_k^c\|$. From (22), and by applying the inequality in Lemma 1 with $0 < (\alpha+1)/2 < 1$ we obtain

$$\begin{aligned}
\dot{V} &\leq -\epsilon \sum_{k=1}^{m} \sum_{i=1}^{d} \left(|[\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c]_i|^2 \right)^{(\alpha+1)/2} \\
&\leq -\epsilon \sum_{k=1}^{m} \left(\sum_{i=1}^{d} |[\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c]_i|^2 \right)^{(\alpha+1)/2}
\end{aligned}$$

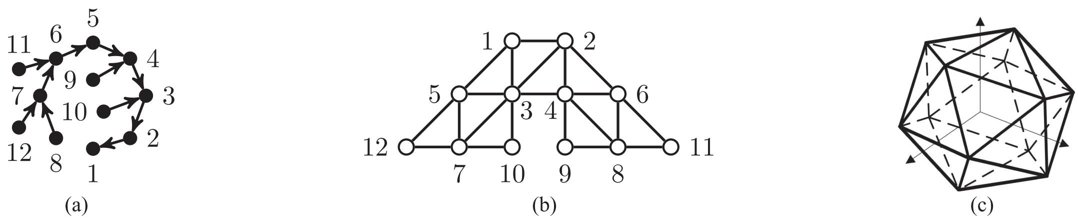


Fig. 2. Formation setup. (a) Orientation sensing graph $\mathcal{G}_o$. (b) Bearing sensing graph $\mathcal{G}_b$. (c) The desired formation shape.

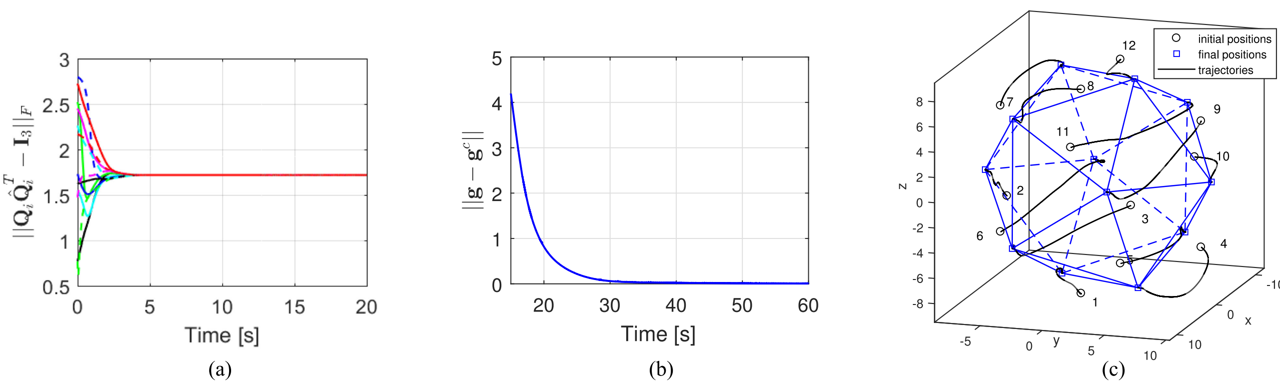


Fig. 3. Simulation 1: Finite-time orientation estimation and finite-time formation control of a 12-agent system with undirected orientation sensing graph. (a) Orientation estimation errors. (b) Bearing error $\|\mathbf{g}-\mathbf{g}^c\|$ versus time. (c) Trajectories of all agents.

$$\begin{aligned}
&\leq -\epsilon \sum_{k=1}^{m} \left(\|\mathbf{Q}^c \mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c\|^2\right)^{(\alpha+1)/2} \\
&\leq -\epsilon \sum_{k=1}^{m} \left(\|\mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c\|^2\right)^{(\alpha+1)/2} \\
&\leq -\epsilon \left(\sum_{k=1}^{m} \|\mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c\|^2\right)^{(\alpha+1)/2}.
\end{aligned} \tag{24}$$

The inequality (24) can be further derived as

$$\begin{aligned}
\dot{V} &\leq -\epsilon \left(\sum_{k=1}^{m} \mathbf{g}_k^{c\,T} \mathbf{P}_{\mathbf{g}_k} \mathbf{g}_k^c\right)^{(\alpha+1)/2} \\
&\leq -\epsilon \left(\sum_{k=1}^{m} \mathbf{g}_k^{T} \mathbf{P}_{\mathbf{g}_k^c} \mathbf{g}_k\right)^{(\alpha+1)/2} \\
&\leq -\epsilon \left(\sum_{k=1}^{m} \frac{1}{\|\mathbf{z}_k\|^2} \mathbf{z}_k^{T} \mathbf{P}_{\mathbf{g}_k^c} \mathbf{z}_k\right)^{(\alpha+1)/2} \\
&\leq -\frac{\epsilon}{(2s\sqrt{n-1})^{\alpha+1}} \left(\sum_{k=1}^{m} \mathbf{z}_k^{T} \mathbf{P}_{\mathbf{g}_k^c} \mathbf{z}_k\right)^{(\alpha+1)/2} \\
&\leq -\gamma \left(\sum_{k=1}^{m} \mathbf{z}_k^{T} \mathbf{P}_{\mathbf{g}_k^c} \mathbf{z}_k\right)^{(\alpha+1)/2}
\end{aligned} \tag{25}$$

where $\gamma = \epsilon/(2s\sqrt{n-1})^{\alpha+1}$. Rewriting (25) as

$$\begin{aligned}
\dot{V} &\leq -\gamma \left[\sum_{k=1}^{m} (\mathbf{z}_k - \mathbf{z}_k^c)^T \mathbf{P}_{\mathbf{g}_k^c} (\mathbf{z}_k - \mathbf{z}_k^c)\right]^{(\alpha+1)/2} \\
&\leq -\gamma \left[(\mathbf{p}-\mathbf{p}^c)^T \bar{\mathbf{H}}^T \operatorname{diag}(\mathbf{P}_{\mathbf{g}_k^c}) \bar{\mathbf{H}} (\mathbf{p}-\mathbf{p}^c)\right]^{(\alpha+1)/2} \\
&\leq -\gamma \left[\boldsymbol{\delta}^T \tilde{\mathbf{R}}^T(\mathbf{p}^c) \tilde{\mathbf{R}}(\mathbf{p}^c) \boldsymbol{\delta}\right]^{(\alpha+1)/2}
\end{aligned} \tag{26}$$

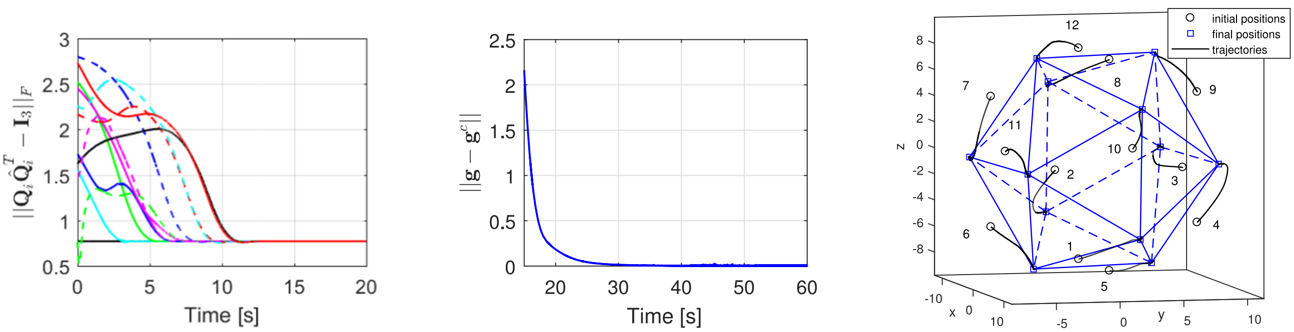
where $\tilde{\mathbf{R}}(\mathbf{p}^c) = \operatorname{diag}(\mathbf{P}_{\mathbf{g}_k^c})\bar{\mathbf{H}}$. Under the assumption that the desired formation is infinitesimally bearing rigid, the matrix $\mathbf{M}^c = \tilde{\mathbf{R}}^T(\mathbf{p}^c)\tilde{\mathbf{R}}(\mathbf{p}^c)$ has $d+1$ zero eigenvalues. Since $\boldsymbol{\delta} \perp \operatorname{Range}(\mathbf{1}_n \otimes \mathbf{I}_d)$, and $\tilde{\mathbf{R}}(\mathbf{p}^c)$ and $\mathbf{R}(\mathbf{p}^c)$ have the same rank and null space, i.e., $\mathcal{N}(\tilde{\mathbf{R}}^T(\mathbf{p}^c)\tilde{\mathbf{R}}(\mathbf{p}^c)) = \mathcal{N}(\mathbf{R}(\mathbf{p}^c)) = \operatorname{span}\{\operatorname{Range}(\mathbf{1}_n \otimes \mathbf{I}_d), \mathbf{r}^c\}$. Let ϕ is the angle between $\boldsymbol{\delta}$ and $-\mathbf{r}^c$, then we obtain

$$\boldsymbol{\delta}^T \tilde{\mathbf{R}}^T(\mathbf{p}^c)\tilde{\mathbf{R}}(\mathbf{p}^c)\boldsymbol{\delta} \geq \lambda_{d+2}\|\boldsymbol{\delta}\|^2 \sin^2\phi \geq \lambda_{d+2}\|\boldsymbol{\delta}\|^2 \sin^2\phi_0$$

where λ_{d+2} is the smallest nonzero eigenvalue of $\mathbf{M}^c$, and $\phi_0 \leq \phi(t)$, $\phi(t) \in [0, \pi/2)$, due to $\boldsymbol{\delta} \to \mathbf{0}$ as $t \to \infty$ as interpreted in Fig. 1. Substituting the aforementioned inequality into (26) yields

$$\dot{V} \leq -\gamma \left(\lambda_{d+2}\|\boldsymbol{\delta}\|^2 \sin^2\phi_0\right)^{(\alpha+1)/2} \leq -\kappa V^{(\alpha+1)/2} \tag{27}$$

where $\kappa = \gamma(2\lambda_{d+2}\sin^2\phi_0)^{\frac{\alpha+1}{2}}$. For $0 < \alpha < 1$, we have $\frac{1}{2} < \frac{\alpha+1}{2} < 1$. It follows from Lemma 2 that $V \to 0$ in finite time if initially we have $\mathbf{p}(0) \neq \mathbf{p}\prime$. Combining this result with Theorem 5, we conclude that $\mathbf{p} = \mathbf{p}^c$ is almost globally finite time stable with a settling time $T_c \leq t_c + V(t_c)^{1-\frac{\alpha+1}{2}}/(\kappa(1-\frac{\alpha+1}{2})) = t_c + 2V(t_c)^{\frac{1-\alpha}{2}}/(\kappa(1-\alpha))$. ■



(a) (b) (c)

Fig. 4. Simulation 2: Finite-time orientation estimation and finite-time formation control of a 12-agent system with directed orientation sensing graph given in Fig. 2(a). (a) Orientation estimation errors. (b) Bearing error $\|\mathbf{g} - \mathbf{g}^c\|$ versus time. (c) Trajectories of all agents.

VI. Simulation

Consider a system of twelve agents in 3-D whose bearing sensing graph $\mathcal{G}_b$ is given in Fig. 2(b). The formation control objective of the system is to achieve a *regular icosahedron* shape [see Fig. 2(c)]. The set of desired bearing vectors is derived from a desired configuration, e.g., $\mathbf{g}^*_{ij} = \frac{\mathbf{p}_j - \mathbf{p}_i}{\|\mathbf{p}_j - \mathbf{p}_i\|}$, whose positions of the agents are given as follows: $\mathbf{p}_1 = [1, 0, -\tau]^T$, $\mathbf{p}_5 = [-1, 0, -\tau]^T$, $\mathbf{p}_4 = [0, \tau, -1]^T$, $\mathbf{p}_6 = [0, -\tau, -1]^T$, $\mathbf{p}_2 = [\tau, -1, 0]^T$, $\mathbf{p}_3 = [\tau, 1, 0]^T$, $\mathbf{p}_{10} = [-\tau, 1, 0]^T$, $\mathbf{p}_{11} = [-\tau, -1, 0]^T$, $\mathbf{p}_7 = [0, -\tau, 1]^T$, $\mathbf{p}_9 = [0, \tau, 1]^T$, $\mathbf{p}_8 = [1, 0, \tau]^T$, and $\mathbf{p}_{12} = [-1, 0, \tau]^T$, where $\tau = (1 + \sqrt{5})/2$. In addition, Agents 1,2, and 3 keep on rotating about their local x-, y-, and z-axes with the same angular velocity of 0.15 rad/s, i.e., $\omega_1^1 = [0.15, 0, 0]^T$, $\omega_2^2 = [0, 0.15, 0]^T$, and $\omega_3^3 = [0, 0, 0.15]^T$, respectively; orientations of the other agents are fixed.

Simulation results are provided for a formation of 12 agents under the orientation estimation law (5) and the control law (16) with $\alpha = 1/2$ in two orientation sensing graphs $\mathcal{G}_o$: undirected graph, which has the same architecture as the one given in Fig. 2(b), and a rooted acyclic digraph [see Fig 2(a)]. The formation control loop is activated at the time instant $t = 15\,s$. The simulation results for the two formation control scenarios are shown in Figs. 3 and 4, respectively. It can be observed that orientations of the agents are estimated up to a coordinate rotation, and the agents achieve the desired formation shape in finite time.

VII. Conclusion

In this paper, a finite-time bearing-only formation control strategy via finite-time orientation estimation has been presented. For estimation of orientation, we proposed a finite-time orientation estimation law and established finite-time convergence of the estimated orientations for two classes of orientation sensing graphs: 1) connected undirected graph and 2) rooted acyclic digraph. Through rigorous analyses of almost global stability and finite-time convergence, we proved that under the proposed orientation estimation and formation control schemes the desired formation shape can be almost globally achieved in a finite time. The upper bounds on the convergence time are also estimated.

Some remarks can be made for the future works. The orientation estimation (5) and the formation control (16) might run in parallel. However, in such scenario, the estimated orientation error, i.e., $\mathbf{Q}^c(t), t \le T_o$, in (19) is time varying; thus, it is more challenging to study the behaviors of the system (16), which is left for future work. Moreover, a possible research direction is to investigate finite-time bearing-only formation control of directed formations or for autonomous agents with more general dynamics.

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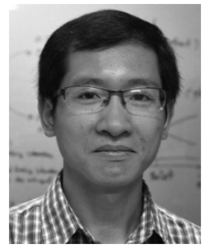

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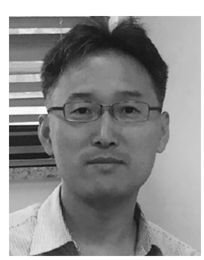

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