Consensus of Higher Order Agents: Robustness and Heterogeneity

Dwaipayan Mukherjee and Daniel Zelazo

***Abstract*—This paper explores the use of Kharitonov's Theorem on a class of linear multiagent systems. First, we study a network of the mth order ($m \geq 2$) linear uncertain interval plants and provide conditions for achieving full-state consensus, which relate the stability margins of each agent to the spectrum of the graph Laplacian. Then, a robustness analysis for such systems is presented when an edge weight in the underlying graph is perturbed. The same Kharitonov-based analysis proves useful in a related problem, where heterogeneous higher order linear models of agents are considered in a setup similar to pinning control, and conditions for consensus among the follower agents are derived. Numerous simulation examples validate the results.**

***Index Terms*—Higher order consensus, Kharitonov's theorem, Laplacian spectra, robust consensus.**

I. Introduction

MULTIAGENT systems have found applications in several problem domains ranging from distributed computing [1], power systems [2], and robotic applications [3] to modeling of social networks [4]. In analyzing the performance of multiagent systems, it is important to ascertain the system's capability in achieving certain basic goals such as consensus or synchronization [5]. This is because several applications, such as formation control of vehicles [6], co-ordinated movements [7], coverage of a certain area [8], etc., can often be addressed using variants of consensus seeking control laws. Hence, the ability of a multiagent system to achieve consensus is at the core of its functionality. It is thus imperative to understand whether a multiagent system can achieve consensus when there are uncertainties in the models of the agents, or in the interaction topology, or both. The uncertainties may result from imprecise mathematical models [9], while perturbations present themselves in the form of external attacks aiming to disrupt the network [10] or antagonistic interactions among agents [11].

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Along this avenue, there have been studies related to robustness of consensus over weighted undirected and directed graphs [12]–[14] where edge weight perturbations were considered. While the properties of the Laplacian matrix, from a graph theoretic perspective, are pivotal to such studies, the effect of edge weight perturbations on the Laplacian spectra, that may lead to phenomenon such as clustering in multiagent systems [15], has also been investigated. However, single, double (see [16], [17] and the references therein), or higher order integrator models for the agents are ubiquitous in such studies, [18], [19] and these higher order integrators have been shown to be adequate to model vehicles [20]. Few works have considered higher order linear models of the individual agents [21], [22], and carried out robustness studies of such systems under additive uncertainties in the agent models. Some others [9] have considered robustness of linear parameter varying agent models. In general, multiagent systems are susceptible to uncertainties or perturbations, both in the agent models, and in the network parameters.

As another direction of work, heterogeneous higher order linear [23]–[25] or nonlinear [26] models of agents have been considered. In these works, and several others [5], [27], [28], there is a local controller for each agent to ensure that the individual agents conform to an *exosystem*. The overall controller comprises a local model/observer-based dynamic controller that makes the agent-local controller dynamics embed an internal model of the same virtual exosystem.

In this paper, we study the robustness properties of consensus-seeking networks where uncertainties exist in the dynamics of the individual agents, and perturbations arise in the edge weights. Thus, we consider a network of systems modeled as linear *interval plants* connected over a weighted graph with perturbations in the edge weights. First, we will try to answer the following question: Suppose a network with a specific structure is designed for a system of identical agents to achieve consensus. Can the edge weights of the network be tuned to still achieve consensus if the physical parameters of the agent models are altered from their nominal values? In this part, we do not consider perturbations on edge weights, rather these weights are design parameters. Subsequently, we determine how much these weights can be perturbed and yet consensus be maintained. Our primary approach to these problems is the application of Kharitonov's theorem on the stability of interval polynomials [29]. We also show that this framework can be applied to consensus problems for networks with higher order linear and *heterogeneous* dynamic agents.

Our first contribution thus considers a network of uncertain, yet homogeneous, higher order linear dynamical systems. Each agent is described by an interval plant, and the agents are assumed to interact over a connected and static network. The main contributions are stated below:

1) We characterize the bound on the ratio between the lower and upper gain margins of interval plants in terms of network parameters.
2) For a consensus-seeking network comprised of interval plants, we obtain bounds on an edge weight perturbation.

We also consider consensus among n heterogeneous agents. Unlike much of the literature dealing with synchronization of such agents, we do not use *local* dynamic controllers. Our contribution in this direction may be summarized as follows:

3) For a network of heterogeneous linear agents, we propose a pinning control-based strategy involving a choice of network parameters and a dynamical leader model to ensure full-state consensus in the followers.

Although this problem is different from the robustness question we first address, by appropriate modeling it can also be approached using Kharitonov's Theorem.

The paper is organized in the following manner. Section II reviews some basics from interval plants, and Kharitonov's theorem. The two problems pertaining to homogeneous agent dynamics are defined and solved in Section III. Consensus of heterogeneous higher order agents is addressed in Section IV. Simulations are provided in Section V to demonstrate the results. Finally, Section VI offers some concluding remarks.

Preliminaries: We employ standard notions from algebraic graph theory [30]. We consider weighted, undirected graphs, denoted by the triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, where $\mathcal{V}$ is the node set, $\mathcal{E}$ the edge set, and $W \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{E}|}$ the diagonal weight matrix. We make extensive use of the incidence matrix of $\mathcal{G}$, denoted $E(\mathcal{G})$, and defined such that $[E(\mathcal{G})]_{ij} = 1$ if node i is the initial node of edge e_k, $[E(\mathcal{G})]_{ij} = -1$ if node i is the terminal node of e_k, and $[E(\mathcal{G})]_{ij} = 0$ otherwise. The graph Laplacian of $\mathcal{G}$ is defined as $\mathcal{L} = E(\mathcal{G})WE(\mathcal{G})^T$ [30], and the edge Laplacian as $\mathcal{L}_e = E(\mathcal{G})^T E(\mathcal{G})W$ [16]. We denote by $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}_e$ the graph and edge Laplacian matrices with unit weights (i.e., $W = I$), respectively.

We now describe some special factorizations of the Laplacian matrices, using the notions of spanning trees. Using [12, Proposition II.1] for a connected graph $\mathcal{G}$, it follows that the weighted graph Laplacian matrix $\mathcal{L}$ is similar to

$$\begin{bmatrix} E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau) RWR^T & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}$$

where $E(\mathcal{G}_\tau)$ is the incidence matrix for a spanning tree $\mathcal{G}_\tau \subseteq \mathcal{G}$, the matrix R is given by

$$E(\mathcal{G}) = [E(\mathcal{G}_\tau)\, E(\mathcal{G}_c)] = E(\mathcal{G}_\tau)[I_{n-1}\, T_\tau] = E(\mathcal{G}_\tau)R \quad (1)$$

with $T_\tau = (E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{-1} E(\mathcal{G}_\tau)^T E(\mathcal{G}_c)$, [16]. Here, $\mathcal{G}_c \subset \mathcal{G}$ is the cotree of $\mathcal{G}_\tau$. Thus, nonzero eigenvalues of $\mathcal{L}$ are identical to those of $E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau) RWR^T$.

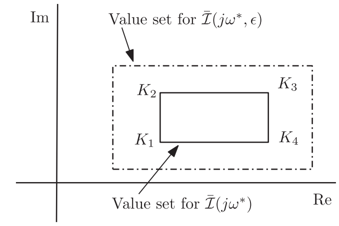

Fig. 1. Value sets at a fixed ω^* for $\bar{\mathcal{I}}(s)$ and $\bar{\mathcal{I}}(s, \epsilon)$.

II. Interval Plants

One approach to modeling uncertain linear dynamical systems assumes that the parameters of the system may vary within a known interval. Equivalently, the coefficients of the characteristic polynomial of the system have unknown coefficients that lie in a specified interval; such polynomials are called *interval polynomials* and are of the form $\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^m$, where the coefficients lie in the interval $\delta_i \in (\underline{\delta}_i, \bar{\delta}_i) \subset \mathbb{R}$, $i = 1, 2, \ldots, m$. These interval polynomials give rise to *interval plants*, whose transfer functions are the ratios of two such interval polynomials [29]. An important result concerning the stability of interval polynomials is Kharitonov's Theorem.

Theorem 1 ([29]): Let $\mathcal{I}(s)$ be the set of real polynomials of degree m given by $\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_m s^m$, where the coefficients lie in the interval $\delta_i \in (\underline{\delta}_i, \bar{\delta}_i) \subset \mathbb{R}$, $i = 1, 2, \ldots, m$. Every polynomial in $\mathcal{I}(s)$ is Hurwitz if and only if the following four extreme polynomials are Hurwitz:

$$K_1(s) = \underline{\delta}_0 + \underline{\delta}_1 s + \bar{\delta}_2 s^2 + \bar{\delta}_3 s^3 + \underline{\delta}_4 s^4 + \underline{\delta}_5 s^5 + \bar{\delta}_6 s^6 + \cdots$$
$$K_2(s) = \underline{\delta}_0 + \bar{\delta}_1 s + \bar{\delta}_2 s^2 + \underline{\delta}_3 s^3 + \underline{\delta}_4 s^4 + \bar{\delta}_5 s^5 + \bar{\delta}_6 s^6 + \cdots$$
$$K_3(s) = \bar{\delta}_0 + \bar{\delta}_1 s + \underline{\delta}_2 s^2 + \underline{\delta}_3 s^3 + \bar{\delta}_4 s^4 + \bar{\delta}_5 s^5 + \underline{\delta}_6 s^6 + \cdots$$
$$K_4(s) = \bar{\delta}_0 + \underline{\delta}_1 s + \underline{\delta}_2 s^2 + \bar{\delta}_3 s^3 + \bar{\delta}_4 s^4 + \underline{\delta}_5 s^5 + \underline{\delta}_6 s^6 + \cdots .$$

For a fixed $s = j\omega^*$, the set of values assumed by the polynomials in $\mathcal{I}(j\omega^*)$ form a rectangle in the complex plane, with vertices at $\{K_i(j\omega^*)\}_{i=1,\ldots,4}$. This rectangle is also known as the *value set*. Similarly, we define families of monic interval polynomials, denoted by $\bar{\mathcal{I}}(s, \epsilon)$, for $\epsilon \geq 0$, whose value sets are shown in Fig. 1. This set contains polynomials given by $\delta_\epsilon(s) := [\underline{\delta}_0 - \epsilon, \bar{\delta}_0 + \epsilon] + \cdots + [\underline{\delta}_{m-1} - \epsilon, \bar{\delta}_{m-1} + \epsilon]s^{m-1} + s^m$. For notational simplicity, we denote $\bar{\mathcal{I}}(s, 0) \equiv \bar{\mathcal{I}}(s)$.

Remark 1: If every polynomial in $\bar{\mathcal{I}}(s, \epsilon)$ is Hurwitz, for some $\epsilon > 0$, then there exists a polynomial $p_\epsilon(s) = \epsilon + \cdots + \epsilon s^{m-1}$, such that for any $\delta(s) \in \bar{\mathcal{I}}(s)$, $\delta(s) \pm p_\epsilon(s)$ is Hurwitz.

III. Consensus Among Homogeneous Linear Agents

We are interested in reaching full-state consensus among a network of linear dynamical systems of order m that interact over an information exchange network modeled by a weighted and undirected graph, $\mathcal{G}$. For every pair of agents i, j,

$$\lim_{t\to\infty} \|x_i^{(k)}(t) - x_j^{(k)}(t)\| = 0, \ k = 0, \ldots, m-1$$

is required, where $x_i^{(k)}(t)$ is the kth time derivative of the state $x_i(t)$ of agent i. We assume a feedback of the form

$$u_i = \sum_{\ell=0}^{m-1} \beta_\ell \sum_{j\in\mathcal{N}_i} w_{ij}(x_j^{(\ell)} - x_i^{(\ell)}). \tag{2}$$

The coefficients β_ℓ and the edge weights w_{ij} must be designed to ensure that full-state consensus is achieved.

We consider all the agents to have identical, but unknown dynamics and model the agents as a family of interval plants,

$$x_i^{(m)} + \alpha_{m-1}x_i^{(m-1)} + \cdots + \alpha_0 x_i = u_i \tag{3}$$

where the coefficients, $\alpha_j, j = 0, 1, \ldots, m-1$, belong to a real interval, $\alpha_j \in [\underline{\alpha}_j, \bar{\alpha}_j]\,\forall j$. Such plants are used in mechanical systems, such as robotic manipulators, and controllers for such systems have been designed using Kharitonov's Theorem [31].

Remark 2: A linear, controllable SISO system has a state space representation that is equivalent to the the model in (3). Even if the state space representation is not in the controller/controllable canonical forms, we can always find a suitable similarity transformation to transform the system to the canonical form due to controllability. Consensus in the original states of the agents, say $x_i^{(k)}$, is equivalent to that in the transformed states. An example of such a system is the cart and double inverted pendulum in [32, Problem 10.52].

We will consider agents having linear, uncertain dynamics of order $m \geq 2$ and obtain conditions for consensus. Later, we study the effect of perturbing nominal edge weights.

A. Uncertain Higher Order Agents: Consensus Design

In studying the uncertainty in agent dynamics vis-à-vis and the properties of the underlying network, we make the following assumption to ensure that the states are bounded at consensus as in [21]. If boundedness is not required, this assumption may be removed.

Assumption 1: There exists an $\epsilon > 0$ such that the family of monic interval polynomials $\bar{\mathcal{I}}(s, \epsilon)$ is Hurwitz.

Remark 3: If the exact models of the dynamic systems are not precisely known, it is not too restrictive to assume that their dynamics are nevertheless stable. The ϵ-refinement adds a certain margin of stability for the entire uncertain family as illustrated by the value set for a given frequency ω^* in Fig. 1.

The problem we first study is stated below.

Problem 1: Consider a set of n identical, but uncertain, dynamical systems modeled by the interval plant (3) that interact over an undirected and weighted graph $\mathcal{G}$. Design the edge weights of the graph such that the agents achieve full-state consensus.

The problem of designing edge weights occurs commonly in the context of the consensus problem, such as in [33] and [34]. The choice of parameters $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ in (2) may also be a part of the design in Problem 1, which we discuss later.

Similar to [21], we want all the agents to converge to the consensus state $\xi(t)$, which evolves according to the dynamics $\xi^{(m)} + \alpha_{m-1}\xi^{(m-1)} + \cdots + \alpha_0\xi = 0$ with coefficients $\alpha_j \in [\underline{\alpha}_j, \bar{\alpha}_j]\,\forall j$ as in (3). The consensus dynamics may thus be stable or unstable depending on the stability of the agents' dynamics.

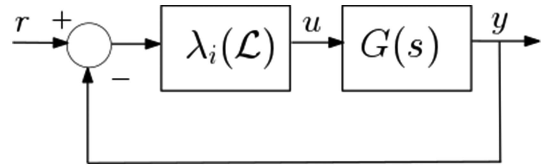

Fig. 2. Polynomial $\pi_i(s)$ equivalently represented as the characteristic polynomial of a closed-loop feedback system.

If Assumption 1 holds, then the desired consensus dynamics is stable with bounded states.

We stack all the agents' states into the vector $\boldsymbol{x} = [(x^{(0)})^T (x^{(1)})^T \cdots (x^{(m-1)})^T]^T$, where $x^{(k)} = [x_1^{(k)} \cdots x_n^{(k)}]^T$. Plugging in the control law in (2), the system is described as

$$\dot{\boldsymbol{x}} = \bar{\boldsymbol{A}}\boldsymbol{x}, \quad \bar{\boldsymbol{A}} = \begin{bmatrix} 0 & \boldsymbol{I}_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{I}_n \\ \boldsymbol{\Lambda}_0 & \boldsymbol{\Lambda}_1 & \cdots & \boldsymbol{\Lambda}_{m-1} \end{bmatrix} \tag{4}$$

where $\boldsymbol{\Lambda_j} = -\alpha_j \boldsymbol{I}_n - \beta_j\mathcal{L}$ and $\mathcal{L}$ is the weighted graph Laplacian of the undirected graph $\mathcal{G}$. The characteristic equation for the system in (4) is given by

$$P(s) = \det\left[s^m \boldsymbol{I}_n + \sum_{j=0}^{m-1}(\alpha_j \boldsymbol{I}_n + \beta_j\mathcal{L})s^j\right] = 0. \tag{5}$$

From [21, Lemma 4], it follows that the characteristic polynomial $P(s)$ in (5) can be expressed as

$$P(s) = \prod_{i=1}^{n}\left[s^m + \sum_{j=0}^{m-1}(\alpha_j + \beta_j\lambda_i(\mathcal{L}))s^j\right] \tag{6}$$

where $\lambda_i(\mathcal{L})$ is the ith eigenvalue of the graph Laplacian $\mathcal{L}$. For full-state consensus, the polynomial $\bar{P}(s)$ is given by

$$\bar{P}(s) = \prod_{i=2}^{n}\left[s^m + \sum_{j=0}^{m-1}(\alpha_j + \beta_j\lambda_i(\mathcal{L}))s^j\right] = \prod_{i=2}^{n}\pi_i(s)$$

needs to be Hurwitz [21].

Each of the polynomials $\pi_i(s)$, corresponding to an eigenvalue $\lambda_i(\mathcal{L})$, can be interpreted as the characteristic polynomial of a closed-loop proportional-gain feedback system, as in Fig. 2. The interval plant $G(s)$ can be given by

$$G(s) = \frac{\beta_{m-1}s^{m-1} + \beta_{m-2}s^{m-2} + \cdots + \beta_0}{s^m + \alpha_{m-1}s^{m-1} + \cdots + \alpha_0} \tag{7}$$

and the proportional gain is then the corresponding eigenvalue of the Laplacian $\lambda_i(\mathcal{L})$. Thus, each of the nonzero Laplacian eigenvalues can be viewed as a feedback gain that must stabilize $G(s)$ to ensure consensus. This is tantamount to meeting $n-1$ design conditions, derived from $n-1$ linear systems. The denominator of (7) is an interval polynomial, while the numerator is a polynomial with constant coefficients. In some cases, the choice of $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ may also be a part of the design problem.

Suppose Assumption 1 does not hold. Since the open-loop zeros of $G(s)$ cannot be placed arbitrarily by choosing

$\{\beta_\ell\}_{\ell=0,\ldots,m-1}$, $G(s)$ may have both open-loop poles and zeros in the right-half plane (rhp). If such a plant $G(s)$ is stabilizable by some proportional gain $k > 0$, then we must have $k \in (\underline{k}, \bar{k}) \subset \mathbb{R}_+$. Hence, $\bar{P}(s)$ is Hurwitz when each $\lambda_i(\mathcal{L})$ (serving as k for some $\pi_i(s)$) belongs to $(\underline{k}, \bar{k})$.

Remark 4: If Assumption 1 is lifted but $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ is such (by designer's choice or as part of the problem set-up) that $G(s)$ has all its open-loop zeros in the left-half plane (lhp), then $0 < \underline{k}$ and $\bar{k} \to \infty$. Similarly, if Assumption 1 holds and $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ results in $G(s)$ having open-loop zeros in the right-half plane, then $\bar{k} < \infty$ and $\underline{k} = 0$.

If the most stringent requirements hold, where $G(s)$ has some of its poles and zeros in the rhp, there may exist a finite interval, $(\underline{k}, \bar{k}) \subset \mathbb{R}_+$, such that if all the nonzero eigenvalues of $\mathcal{L}$ belong to this interval, $\bar{P}(s)$ is Hurwitz. The problem of designing a suitable network becomes equivalent to one of ensuring that the $n-1$ nonzero eigenvalues of $\mathcal{L}$ are confined to a finite interval on $\mathbb{R}_+$. Since α_i vary independently, the required interval for the Laplacian eigenvalues can be obtained by applying Kharitonov's Theorem on $K(s)$, given by

$$K(s) = s^m + \chi_{m-1}s^{m-1} + \cdots + \chi_1 s + \chi_0, \tag{8}$$

where $\chi_i \in [\underline{\alpha}_i + k\beta_i, \bar{\alpha}_i + k\beta_i]$, for $i = 0, 1, \ldots, m-1$.

Thus, we reduce the problem of computing $n-1$ eigenvalues to verify the stability of four polynomials, using Theorem 1. If Kharitonov's Theorem on (8) specifies a range of real values, $(\underline{k}, \bar{k})$, for $\lambda_i(\mathcal{L})_{i=2,\ldots,n}$, then the closed loop system in Fig. 2 is robustly stable. This leads to the following lemma.

Lemma 1: Suppose the interval plant $G(s)$ in (7) with fixed $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ is robustly stabilized by a proportional gain $k \in (\underline{k}, \bar{k})$. If all nonzero eigenvalues of the Laplacian $\mathcal{L}$ belong to the interval $(\underline{k}, \bar{k})$, then the multiagent system in (3) under control law (2) achieves consensus in its states.

Proof: See Appendix. ■

The following example elucidates the preceding discussion.

Example 1: Consider the uncertain dynamics of n identical agents described by the equation $x^{(4)} + \alpha_3 x^{(3)} + \alpha_2 x^{(2)} + \alpha_1 x^{(1)} + \alpha_0 x = u$, where $\alpha_0 \in [0.90, 1.10]$, $\alpha_1 \in [3.28, 3.60]$, $\alpha_2 \in [4.20, 4.68]$, and $\alpha_3 \in [3.0, 3.28]$. Further, suppose the set $\{\beta_l\}_{l=0,\ldots,3}$ is given by $\{-0.02525, 25.249, 0.999, 1\}$. In order to obtain the allowable range of k, we use the Routh tabulation on the four polynomials given by Theorem 1. This leads to the corresponding ranges for k as $(0, 0.46) \cup (17.6, 43.56)$, $(0, 0.68) \cup (16.6, 35.64)$, $(0, 0.55) \cup (17.22, 43.56)$, and $(0, 0.57) \cup (16.99, 35.64)$, respectively. Upon taking the intersection over these four ranges, we may ascertain that for $k \in (0, 0.46) \cup (17.6, 35.64)$, the entire interval family is stabilized.

We now discuss how the designer may choose the set $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ to ensure consensus. If Assumption 1 holds, there always exists a choice $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ such that the closed loop system in Fig. 2 is stable for some $\lambda_i(\mathcal{L}) > 0$ [and consequently for some finite range of $\lambda_i(\mathcal{L})$]. Even for the given choice of $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ in Example 1, if all nonzero eigenvalues of the graph Laplacian lie in the interval $(0, 0.46)$, consensus will be achieved. This is also due to Assumption 1 which ensures that the uncertain agent dynamics are nevertheless stable. If we fix up the leading coefficient β_{m-1}, and the zeros of $G(s)$ at $z_1, z_2, \ldots, z_{m-1}$ so that the four extremal systems corresponding to $G(s)$ are closed-loop stable for a desired range of gains, given by an interval $\mathcal{P}_s$ (suppose the network is such that $\lambda_i(\mathcal{L}) \in \mathcal{P}_s$ $i = 2, \ldots, n$), then we can choose $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ by equating coefficients of like terms in $\beta_{m-1}\prod_{i=1}^{m-1}(s - z_i) = \beta_{m-1}s^{m-1} + \beta_{m-2}s^{m-2} + \cdots + \beta_0$. When Assumption 1 holds, it suffices to choose $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ such that the zeros of $G(s)$ are all in the lhp. Conversely, for a *well chosen* set $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$, it may be ensured that for any positive proportional gain, $k = \lambda_i(\mathcal{L})$, $G(s)$ will be stable in closed loop. However, even if Assumption 1 is relaxed, it may be possible to design a suitable network to achieve consensus (although the states may not be bounded at consensus) provided the plant $G(s)$ is stabilizable with a proportional gain.

Remark 5: In Example 1, we have considered the parameters $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ to be fixed by the problem setup. However, in case the designer has freedom to choose these parameters, the root locus technique may be applied in choosing them, to place open-loop zeros suitably, corresponding to the poles of the four extremal polynomials. If Assumption 1 does not hold, the choice of parameters $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ assumes even greater significance because it must be chosen to place the open-loop zeros of $G(s)$, so that the allowable range of the stabilizing k (determined as shown in Example 1) is not an empty set. Furthermore, an increase in m will result in an increase in the number of branches of the root locus for $G(s)$. Thus, the determination of allowable range of k may become more involved as the order of the agent dynamics increases, more so if Assumption 1 is removed.

For a given set $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$, the range of stabilizing proportional gains for the interval plant $G(s)$ may include the union of several disjoint positive intervals. However, here we consider any one such interval of stabilizing gains $(\underline{k}, \bar{k})$, though multiple such disjoint intervals may exist. Thus, in Example 1, we may either confine all Laplacian eigenvalues to $(0, 0.46)$ or to $(17.6, 35.64)$ to achieve consensus. This leads to conservatism and, thus, ensures only sufficiency.

Corollary 1: If there are multiple disjoint intervals $\mathcal{P}_s = (\underline{k}_s, \bar{k}_s)$ (finite or infinite) such that the closed-loop interval plant in Fig. 2 is stable for $\lambda_i(\mathcal{L}) \in \mathcal{P}_s$ $\forall i \neq 1$, for consensus it suffices that the nonzero eigenvalues of $\mathcal{L}$ belong to $\cup_s \mathcal{P}_s$.

Proof: Suppose the nonzero eigenvalues of $\mathcal{L}$, $\lambda_i(\mathcal{L})_{i=2,\ldots,n}$ belong to $\cup_s \mathcal{P}_s$. Then, every polynomial of the form $s^m + \sum_{j=0}^{m-1}(\alpha_j + \beta_j\lambda_i(\mathcal{L}))s^j$, $i = 2, \ldots, n$ is Hurwitz. Thus, the polynomial $\bar{P}(s)$ in (6) is Hurwitz, leading to consensus. ■

Corollary 2: If $\underline{k} < 0$, and $\bar{k} > 0$, then for consensus it suffices to choose positive edge weights that ensure $\lambda_n(\mathcal{L}) < \bar{k}$, and such a choice of edge weights always exists.

Proof: If the range of stabilizing gain is given by $k \in (\underline{k}, \bar{k})$, where $\underline{k} < 0 < \bar{k}$, then for $i = 2, \ldots, n$, if $0 < \lambda_i(\mathcal{L}) < \bar{k}$, $\bar{P}(s)$ will be Hurwitz, leading to consensus. Finally, observe that choosing the weights $W = \mu I$ for $0 < \mu < \bar{k}(\lambda_n(\hat{\mathcal{L}}))^{-1}$ guarantees that $\lambda_n(\mathcal{L}) < \bar{k}$. ■

Theorem 2: Assume that the agents described by the interval plant (3) can be robustly stabilized with any proportional gain $k \in (\underline{k}, \bar{k})$ ($\underline{k} > 0$). Then, using the consensus feedback (2) can achieve full-state consensus for a given choice of $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ if

$$\frac{\underline{k}}{\bar{k}} < \frac{\lambda_{\min}(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))\lambda_{\min}(RR^T)}{\lambda_{\max}(\hat{\mathcal{L}})}. \quad (9)$$

Proof: Suppose the weight on each edge is $\mu > 0$. Define $M = M^T > 0$ such that $M^2 = E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau)$. Applying the similarity transformation $M^{-1}(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))RWR^T M$, it follows that matrix $E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau)RWR^T$ is similar to $(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{\frac{1}{2}} RWR^T (E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{\frac{1}{2}}$. Consequently, the nonzero eigenvalues of $\mathcal{L}$ are greater than a positive real number $\underline{k}$ if and only if the linear matrix inequality $MRWR^T M - \underline{k}\boldsymbol{I}_{n-1} > 0$ holds. Similarly, $E(\mathcal{G})WE(\mathcal{G})^T - \bar{k}\boldsymbol{I}_n < 0$ ensures that the largest eigenvalue of $\mathcal{L}$ is less than $\bar{k}$. Also, since $MRWR^T M > 0$ is symmetric positive definite, its eigenvalues are identical to its singular values. Thus, the lower bound is written as

$$\mu\sigma_{\min}(MRR^T M) > \underline{k} \Rightarrow \mu > \underline{k}\sigma_{\max}([MRR^T M]^{-1}).$$

Since $\|\mathcal{L}\|_2 = \sigma_{\max}(\mathcal{L})$, using submultiplicativity of norms, we may write

$$\underline{k}\sigma_{\max}([MRR^T M]^{-1}) = \underline{k}\sigma_{\max}(M^{-1}(RR^T)^{-1}M^{-1})$$
$$\leq \underline{k}\sigma^2_{\max}(M^{-1})\sigma_{\max}([RR^T]^{-1}) = \frac{\underline{k}}{\sigma^2_{\min}(M)\sigma_{\min}([RR^T])}.$$

Since, $\sigma^2_{\min}(M) = \sigma_{\min}(M^2)$ due to symmetry of M, a conservative sufficiency condition can be further obtained as

$$\mu > \frac{\underline{k}}{\sigma_{\min}(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))\sigma_{\min}(RR^T)}.$$

Further, $\mu\sigma_{\max}(E(\mathcal{G})E(\mathcal{G})^T) < \bar{k}$ needs to be satisfied. Hence, if there exists some real μ such that

$$\frac{\underline{k}}{\sigma_{\min}(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))\sigma_{\min}(RR^T)} < \mu < \frac{\bar{k}}{\sigma_{\max}(\hat{\mathcal{L}})}$$

then the consensus protocol (2) can be designed. This is possible if condition (9) holds. ■

From Theorem 2, it is evident that the feasibility and flexibility of the design depend on the kind of tree that exists in the graph $\mathcal{G}$. Theorem 1 in [35] and the results in [36] imply that the Fiedler eigenvalue of a connected graph on n vertices is at least the same as that for the path graph P_n. Thus, P_n has the lowest value of $\sigma_{\min}(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))$ among all spanning trees on n vertices. On the other hand, a star graph S_n has the maximum value of Fiedler eigenvalue among all spanning trees (see [37, Th. 5.9]). Another interesting class of graphs are the regular graphs, which also includes the complete graph. The above observations motivate a closer inspection of S_n, P_n, cycle graph C_n, and the complete graph K_n.

Corollary 3: Consider n homogeneous agents governed by the uncertain dynamics (3) interacting over a spanning tree $\mathcal{T}_n$, with the control law (2). Then, if

$$\frac{\underline{k}}{\bar{k}} < \frac{\lambda_{\min}(\mathcal{T}_n)^T E(\mathcal{T}_n))}{\lambda_{\max}(\mathcal{T}_n)^T E(\mathcal{T}_n))}$$

the system can achieve full-state consensus. Furthermore, for the star graph $\mathcal{T}_n = S_n$, the bound is $\frac{1}{n}$, and for the path $\mathcal{T}_n = P_n$, this bound is $\tan^2(\pi/2n)$.

Proof: The proof follows from Theorem 2, and the observation for spanning trees, $RR^T = \boldsymbol{I}$. Noting that the spectrum of the star S_n contains one zero eigenvalue, a unit eigenvalue with algebraic multiplicity $n-2$ and an eigenvalue of n with algebraic multiplicity one [38], and that of P_n is given by $\{4\sin^2(\frac{\pi r}{2n})\}_{r=0,\ldots,n-1}$ [38], the proof is completed. ■

A graph may not necessarily be a spanning tree, but contain certain spanning trees as subgraphs. Then, a direct evaluation of its Laplacian spectra is not straightforward. However, based on the structure of the spanning subgraph, certain bounds on the ratio $\underline{k}/\bar{k}$ can be obtained using Theorem 2. If a graph $\mathcal{G}$ has a star spanning tree and the total number of its edges is less than $2n-2$ (see Lemma 6), the right-hand side of the inequality (9) reduces to $\lambda_{\max}(\hat{\mathcal{L}})^{-1}$. The spectral radius of $\hat{\mathcal{L}}(\mathcal{G})$ cannot be greater than that of $\hat{\mathcal{L}}(S_n)$ because we know that the spectral radius of the star S_n and the complete graph K_n (with unit weights on all edges) are the same and no graph can have a greater spectral radius than K_n. Thus, $\underline{k}/\bar{k} < \lambda_{\max}(\hat{\mathcal{L}})^{-1} < 1/n$ is the allowable ratio of the upper and lower gain margins. Thus, as the number of agents n increases, the tolerance for gain margin is tighter if $\underline{k} > 0$. However, an increase in the number of edges does not alter the spectral radius of $\mathcal{L}(\mathcal{G})$, as discussed above, but if the increase in the number of edges (so that $|\mathcal{E}| > 2n-2$) also increases $\sigma_{\min}(RR^T)$ from unity, then the right-hand side of inequality (9) increases, providing greater design flexibility.

For $n > 3$, the upper bound on $\underline{k}/\bar{k}$ in (9) is higher for the star graph than a path graph, because $\tan^2(\pi/2n)$ decays faster than $1/n$ with an increase in n. Consider a graph $\mathcal{G}$ with a spanning path P_n. Assume $|\mathcal{E}| < 2n-2$. Hence, the inequality (9) reduces to

$$\frac{\underline{k}}{\bar{k}} < \frac{4\sin^2(\pi/2n)}{\lambda_{\max}(\hat{\mathcal{L}})} < \frac{4\sin^2(\pi/2n)}{4\cos^2(\pi/2n)} = \tan^2(\pi/2n).$$

As in case of S_n, with an increase in n the ratio of lower to upper gain margins needs to be smaller for P_n, indicating a more stringent stability margin requirement for the uncertain plant. But increasing the number of edges (say, beyond $2n-2$) may increase both the spectral radius $\lambda_n(\hat{\mathcal{L}})$ as well as $\lambda_{\min}(RR^T)$. Hence, a tradeoff is necessary to get more relaxed stability margins. Next, we will consider two types of regular graphs, the cycle C_n and the complete graph K_n.

Corollary 4: Consider n homogeneous agents governed by the uncertain dynamics (3) interacting over a cycle graph C_n with the control law (2). Then, if

$$\frac{\underline{k}}{\bar{k}} < \frac{\lambda_{\min}(E(C_n)^T E(C_n))}{\lambda_{\max}(E(C_n)^T E(C_n))} = \begin{cases} \sin^2(\pi/n), & \text{for even } n \\ 4\sin^2(\pi/2n), & \text{for odd } n \end{cases}$$

the system can achieve full-state consensus.

Proof: Since, the spectra of the cycle graph is given by $\{4\sin^2(\pi r/n)\}_{r=0,\ldots,n-1}$, clearly, the spectral radius for C_n is 4 when n is even and $4\cos^2(\pi/2n)$ when n is odd. For an even number of agents, the limit on $\underline{k}/\bar{k}$ is thus $\sin^2(\pi/n)$ while for an odd number of agents, it is $4\sin^2(\pi/2n)$. ■

Corollary 5: Consider n homogeneous agents governed by the uncertain dynamics (3) interacting over a complete graph K_n with the control law (2). Then, the system can always achieve full-state consensus.

Proof: The spectra of K_n comprises a zero eigenvalue and $n-1$ eigenvalues each equal to n. Thus, the positive spectra of K_n comprises only n with multiplicity $n-1$. Hence, the upper bound on $\underline{k}/\bar{k}$ is unity, which is always satisfied.

Again, we have used explicit knowledge of the Laplacian spectra of K_n instead of Theorem 2. Now, S_n is a spanning tree of K_n with highest algebraic connectivity, equal to unity (follows from application of [37, Th. 5.9], and the Laplacian spectrum for S_n given in [38]). Thus, a limit on the ratio in (9) for K_n, by Theorem 2, is $1/n$. This is because the matrix $T_\tau T_\tau^T$ for a complete graph with a star spanning tree is singular. Thus, vis-à-vis the limit in (9), there is no difference between S_n with $n-1$ edges and K_n with $n(n-1)/2$ edges. But explicit knowledge of the Laplacian spectra indicates otherwise; thus, showing the conservatism of Theorem 2.

Remark 6: Theorem 2 gives a sufficient condition by restricting the eigenvalues of $\mathcal{L}$ to a single interval, whereas, as stated in Corollary 1, there may be multiple intervals $\mathcal{P}_s$, where the gain k may lie to ensure closed-loop stability of the system in Fig. 2. Moreover, the edge weights are identical in the proof of Theorem 2. Thus, certain networks not satisfying Theorem 2 may still achieve consensus due to heterogeneous edge weights. The inequality $\underline{k}\sigma_{\max}(M^{-1}(RR^T)^{-1}M^{-1}) \leq \underline{k}\sigma^2_{\max}(M^{-1})\sigma_{\max}([RR^T]^{-1})$ introduces more conservatism.

The following examples show that Theorem 2 and the subsequent corollaries state sufficient conditions only. Nevertheless, a network that satisfies the condition of Theorem 2 guarantees an *easy* choice of identical edge weights for consensus.

Example 2: Consider the agent models and parameters $\{\beta_l\}_{l=0,\ldots,3}$ as given in Example 1. Suppose the network chosen is S_{40}. Note that for the interval $(17.6, 35.64)$, the condition in Corollary 3 is not satisfied. Thus, it would appear that consensus cannot be achieved. However, note that if we were to confine the nonzero eigenvalues of the Laplacian to the interval $(0, 0.46)$, consensus can be achieved. Now, choosing $\mu = 0.01$ ensures that all nonzero eigenvalues of the Laplacian belong to $(0, 0.46)$. This will naturally lead to slow convergence rates. Alternately, we may also choose $\mu = 0.45$ and in this case there are exactly 38 nonzero eigenvalues equal to 0.45 [which belong to $(0, 0.46)$] and another eigenvalue at 18 [which belongs to $(17.6, 35.64)$], which ensure consensus although the Laplacian spectrum is not confined to any of the two intervals alone.

Example 3: So far only identical edge weights have been considered. However, now consider the same setup as in Example 1 over P_6. It is evident that we cannot allocate all the eigenvalues of the Laplacian in the interval $(17.6, 35.64)$ using identical edge weights because the condition of Corollary 3 is not met with $\underline{k} = 17.6$ and $\bar{k} = 35.64$. However, if the edge weights are chosen as $[0.12\, 0.12\, 0.084\, 0.24\, 9.0]$, the nonzero eigenvalues of the Laplacian are $\{0.033,\ 0.162,\ 0.359,\ 0.451,\ 18.12\}$, which belong to $(0, 0.46) \cup (17.6, 35.64)$ thereby ensuring consensus.

Remark 7: Examples 2 and 3 show that by using $k \in \cup_s \mathcal{P}_s$, as allowable gains for the plant $G(s)$, and also by exploiting heterogeneity in edge weights, the conservatism inherent in Theorem 2 and the subsequent corollaries can be relaxed. However, analyzing the effect of individual edge weights on each of the eigenvalues of the Laplacian is difficult and even in Example 3, for P_6, choosing the edge weights so as to place the eigenvalues in the set $(0, 0.46) \cup (17.6, 35.64)$ is not straightforward (we used trial and error). Thus, for more general and complex networks, the sufficient condition in Theorem 2, wherever satisfied, aids by assigning identical edge weights, in a straightforward manner.

B. Robust Consensus: Edge Weight Perturbation

As a dual to the design problem, consider a multiagent system given by (3) and control law (2) with fixed variables $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$ and possibly heterogeneous edge weights in the network to satisfy Lemma 1.

We now introduce the notion of perturbation in the network through the edge weights. The perturbation of the weight on edge $\{i, j\}$ is modeled as an additive one to the nominal edge weight, given by $w_{ij} + \delta$, with $\delta \in (\underline{\delta}, \bar{\delta})$ for some finite scalars, $\underline{\delta}$, and $\bar{\delta}$. Here, we consider perturbations to a single edge. The uncertainty set (capturing the perturbation) is thus

$$\mathbf{\Delta} = \{\delta \in \mathbb{R} \,:\, \delta \in (\underline{\delta}, \bar{\delta})\}. \tag{10}$$

The perturbed edge weight matrix thus takes the form $W_\Delta = W + \delta P_r P_r^T$, where P_r is the rth standard basis vector in $\mathbb{R}^{|\mathcal{E}|}$ and specifies the edge where the perturbation in weight occurs. The robustness question may now be stated.

Problem 2: For a multiagent system given by (3) and (2), subject to Assumption 1, and satisfying the condition in Lemma 1, obtain the robust stability margin of the consensus protocol to a perturbation δ in edge weight w_{ij}.

The gain margin of the system in (3) or (7) is described by an interval of the form $(\underline{k}, \bar{k})$. This interval could be infinite, in which case, the constraint on the Laplacian eigenvalues is less restrictive. However, for a finite interval (that is, $\underline{k}$ and $\bar{k}$ are both finite), we require the nonzero eigenvalues of the graph Laplacian to belong to the same interval. The required condition for consensus may be stated as follows.

Lemma 2: If the two matrix inequalities given by

$$QWQ^T - \underline{k}\boldsymbol{I}_{n-1} > 0 \tag{11}$$

$$E(\mathcal{G})WE(\mathcal{G})^T - \bar{k}\boldsymbol{I}_n < 0 \tag{12}$$

are satisfied, where $Q = MR = (E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{\frac{1}{2}} R$, and $G(s)$ in (7) can be stabilized with a proportional gain $k \in (\underline{k}, \bar{k})$ in closed loop, then the agents described by the interval plant (3), and driven by the control law (2) achieve consensus.

Proof: Proof follows from using Lemma 1 and the part of the proof of Theorem 2 that establishes the similarity of matrices $E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau) RWR^T$ and $(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{\frac{1}{2}} RWR^T (E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))^{\frac{1}{2}}$. ■

Suppose the conditions (11) and (12) are satisfied by the system with nominal edge weights. It is desired to obtain the limit on the perturbation of a single edge weight so that any one of the inequalities in Lemma 2 is violated. A positive perturbation may violate (12) while a negative perturbation may violate (11). We now endeavor to obtain a limit on the edge weight perturbation to maintain consensus. First, we shall consider agents modeled by uncertain second-order dynamics where the conditions for robust stability are expected to be simpler.

1) Second-Order Agents ($m = 2$): For second-order agents with $m = 2$, the choice of variables β_1 and β_0 and the edge weights are greatly simplified, since any positive real value of β_1 and β_0 ensures consensus for a positive semidefinite Laplacian. Thus, the requirement for consensus is tantamount to the stability of the $n-1$ polynomials given by

$$s^2 + [\underline{\alpha}_1, \bar{\alpha}_1]s + [\underline{\alpha}_0, \bar{\alpha}_0] + \lambda_i(\mathcal{L})(\beta_1 s + \beta_0) \qquad (13)$$

where $\lambda_i(\mathcal{L})$ is the ith nonzero eigenvalue of $\mathcal{L}$. Since a second-order system has infinite upper gain margin, the constraint (12) is redundant. Thus, only a negative perturbation can possibly disrupt the consensus in this case.

Theorem 3: Consider a collection of n second-order systems with dynamics (3) ($m = 2$) implementing the consensus control (2) such that (11) and (12) are satisfied. Then, the system is robustly stable against any perturbation to a single edge weight w_{ij} satisfying

$$\delta > -\frac{1}{P_r^T Q^T (QWQ^T + \gamma \boldsymbol{I}_{n-1})^{-1} Q P_r} \qquad (14)$$

where P_r is the rth standard basis vector in $\mathbb{R}^{|\mathcal{E}|}$ corresponding to the perturbed edge e_r, and $\gamma = \min(\frac{\alpha_0}{\beta_0}, \frac{\alpha_1}{\beta_1})$.

Proof: Since the second-order polynomial of the form in (13) is stable if the coefficients are positive, it suffices to ensure that $\underline{\alpha}_1 + \lambda_i(\mathcal{L})\beta_1$ and $\underline{\alpha}_0 + \lambda_i(\mathcal{L})\beta_0$ are positive for all $\lambda_i(\mathcal{L}) \neq 0$. Since the agents dynamics are robustly stable by Assumption 1, $\underline{\alpha}_1$ and $\underline{\alpha}_0$ are positive. Also, the variables β_1, β_0 are positive by choice. Thus, if $\lambda_i(\mathcal{L}) + \gamma > 0$ for all i, consensus will be achieved. Alternately, for a stable second-order system, $\underline{k} = -\gamma$ and $\bar{k} \to \infty$ in (11) and (12). This leads to the bound in (14). ■

2) Higher Order Agents ($m > 2$): For higher order agents, merely ensuring the positivity of the coefficients of the characteristic polynomial does not guarantee consensus. The amount of perturbation an edge weight can tolerate is therefore related to some bounds that are not necessarily expressed explicitly in terms of the coefficients of the agents' characteristic polynomial or the members of the set $\{\beta_\ell\}_{\ell=0,\ldots,m-1}$. Thus, the analysis in this case is nontrivial.

Theorem 4: The multiagent system in (2) and (3), that satisfies (11) and (12), is robustly stable against perturbations on an edge w_{ij} for all $\delta \in \Delta = (\underline{\delta}, \bar{\delta})$ with

$$\underline{\delta} = -\frac{1}{P_r^T Q^T (QWQ^T - \underline{k}\boldsymbol{I}_{n-1})^{-1} Q P_r}$$

$$\bar{\delta} = \frac{1}{P_r^T E(\mathcal{G})^T (\bar{k}\boldsymbol{I}_n - E(\mathcal{G}) W E(\mathcal{G})^T)^{-1} E(\mathcal{G}) P_r}$$

where P_r is the rth standard basis vector in $\mathbb{R}^{|\mathcal{E}|}$.

Proof: For the perturbed system to achieve consensus, the requirement is $QW_\Delta Q^T - \underline{k}\boldsymbol{I}_{n-1} > 0$, where $W_\Delta = W - P_r|\delta|P_r^T$ ($\delta < 0$) was defined earlier. Equivalently, the condition reduces to the following LMI:

$$[QWQ^T - \underline{k}\boldsymbol{I}_{n-1}] - QP_r|\delta|P_r^T Q^T > 0.$$

Using the Schur Complement form, this may be written as

$$\begin{bmatrix} |\delta|^{-1} & (QP_r)^T \\ QP_r & QWQ^T - \underline{k}\boldsymbol{I}_{n-1} \end{bmatrix} > 0.$$

Again using the Schur Complement form, this may be rewritten as $|\delta|^{-1} - (QP_r)^T (QWQ^T - \underline{k}\boldsymbol{I}_{n-1})^{-1} QP_r > 0$. Applying a similar argument for a positive edge weight perturbation, the upper bound on δ follows. ■

Unlike the case of single integrators, here we have to deal with both an upper and a lower bound on the edge weight perturbation, given by Theorem 4.

3) Multiple Edge Weight Perturbations: So far, we have considered a single edge weight perturbation. But, when multiple edge weights are perturbed, Theorem 4 is still useful. Broadly speaking, there can be two types of such attacks: those of known magnitudes, and those of unknown magnitudes. While the latter is beyond the scope of this paper, the former may be tackled using Theorem 4. To determine whether attacks of known magnitudes on a given set of q out of $|\mathcal{E}|$ edge weights will disrupt consensus or not, it suffices to employ Theorem 4 as in Algorithm 1. This algorithm not only determines whether the perturbed system achieves consensus (which can be determined by just one iteration of the algorithm), but also tells us how *fragile* the system is to the perturbation on each of the perturbed edges. In other words, even if consensus is achieved by the perturbed system, Algorithm 1 also determines how much more perturbation would have been required on a particular edge weight before consensus would have been disrupted.

Algorithm 1: Perturbations On q Edges.

1: $W = W_{\text{nominal}} + \sum_1^q \delta_i P_i P_i^T$
2: $i = 1, j = 0$.
3: Initialize $W = W - \delta_i P_i P_i^T$. Evaluate $\underline{\delta}_i$ and $\bar{\delta}_i$ with $P_r = P_i$, using Theorem 4.
4: If $\delta_i \in (\underline{\delta}_i, \bar{\delta}_i)$, $j = 1$, else $j = 0$.
5: $W = W + \delta_i P_i P_i^T$.
6: $i = i + 1$.
7: If $i < q + 1$ goto 3.
8: If $j = 1$, 'Consensus'; else 'Consensus not guaranteed'

Consider a labeling of the edges such that the perturbed edges are labeled 1 through q, with $\delta_i, i = 1, \cdots, q$, being the perturbation, $W_{\text{nominal}} \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{E}|}$ the diagonal matrix of nominal edge weights, and P_i the ith standard basis in $\mathbb{R}^{|\mathcal{E}|}$.

If it turns out upon computation that $\underline{\delta_i} > \bar{\delta}_i$, the interval $(\underline{\delta}_i, \bar{\delta}_i)$ is an empty set and consensus is not guaranteed.

IV. Consensus Among Heterogeneous Agents: Pinning Control Approach

So far, uncertainty in agent dynamics have been considered, but the dynamics of the agents were identical. In this section, we consider a variation of the *pinning control* for the synchronization of dynamical systems [39]–[44]. Each agent in the network is characterized by an mth order ODE,

$$x_i^{(m)} + \alpha_{m-1,i} x_i^{(m-1)} + \cdots + \alpha_{0,i} x_i = u_i. \tag{15}$$

Here, we assume that the coefficients for each agent are known but are heterogeneous; that is, $\alpha_{s,i} \neq \alpha_{s,j}$, in general, for any pair i, j and $s = 0, \ldots, m-1$. The pinning agent (the leader) is modeled by a linear system of order $m-1$.

Conditions for achieving consensus among such follower agents, by designing a suitable leader dynamics, and connection topology among followers, are derived. The n followers interact among themselves over an undirected graph $\mathcal{G}$, while a leader agent, labeled L, has a dynamics of order $m-1$. The overall graph is a union of the undirected graph $\mathcal{G}$, and a directed graph with n directed edges between each of the n agents and the leader L. In principle, this is similar to a mixed broadcast and communication environment as in [45] and may occur in case of human swarm interactions.

Next, define $\underline{\alpha}_j = \min_i \alpha_{j,i}$ and $\bar{\alpha}_j = \max_i \alpha_{j,i}$. Now, we revert to the definition of family of interval polynomials $\bar{\mathcal{I}}(s,\epsilon)$ containing polynomials $\alpha_\epsilon(s) := [\underline{\alpha}_0 - \epsilon, \bar{\alpha}_0 + \epsilon] + \cdots + [\underline{\alpha}_{m-1} - \epsilon, \bar{\alpha}_{m-1} + \epsilon] s^{m-1} + s^m$, where $\epsilon > 0$ as described in Assumption 1. Note that in the setup of Section III-A, every agent's characteristic polynomial corresponds to the same point in the value set shown in Fig. 1, whereas in this section the characteristic polynomials correspond to different points in the value set. Although we use the interval family $\bar{\mathcal{I}}(s,\epsilon)$ in our analysis throughout this paper, in this section the coefficients are known precisely for each agent, and $\bar{\mathcal{I}}(s,\epsilon)$ is defined to only aid the analysis and not for robustness studies, while in Section III-A the interval family arises owing to the uncertainty inherent in the agent models themselves. Assumption 1 will also be used in this section, though it will be shown later that it may also be replaced with a weaker assumption. The main problem is now stated.

Problem 3: For the multiagent system (15), subject to Assumption 1, determine if it is possible to choose a suitable dynamics for the leader and edge weights on the network, containing the leader and the follower agents, so that the bounded full-state consensus among followers can be achieved. If so, then under what condition is this achievable?

The following Lemma provides a condition that paves the way toward the solution of Problem 3.

Lemma 3: For the multiagent system with dynamics given in (15) and satisfying Assumption 1, there exists a Hurwitz stable plant with dynamics given by $\xi^{(m)} + \eta_{m-1}\xi^{(m-1)} + \cdots + \eta_0 \xi = 0$ if the following matching condition (**M**) is met, **M**: There exist two sets of real numbers $\{k_0, k_1, \ldots, k_{m-1}\}$ and $\{\eta_0, \eta_1, \ldots, \eta_{m-1}\}$, such that

$$\frac{\eta_{m-1} - \alpha_{m-1,i}}{k_{m-1}} = \frac{\eta_{m-2} - \alpha_{m-2,i}}{k_{m-2}} = \cdots = \frac{\eta_0 - \alpha_{0,i}}{k_0} = \rho_i \quad \forall i. \tag{16}$$

Proof: See Appendix. ■

In some works [5], [27], [28], consensus was achieved in heterogeneous higher order agents using internal model principle or observers which use additional local controllers for each agent. An identical internal model of an exosystem, contained by each agent, generates the consensus trajectories. We do not consider any local controllers and show that subject to a matching condition, as in (16), and some robust stability requirements, consensus can be achieved by the followers through information exchange among each other and a pinning control action of the leader. The matching condition in (16), though restrictive, is thus the price paid for achieving consensus, without local controllers, among heterogeneous agents.

Consider a leader of order $m-1$ given by

$$x_L^{(m-1)} + \frac{k_{m-2}}{k_{m-1}} x_L^{(m-2)} + \cdots + \frac{k_0}{k_{m-1}} x_L = u_L \tag{17}$$

where the input u_L may be chosen as 0. Suppose the control law for agent i is given by

$$u_i = \sum_{l=0}^{m-1} \sum_{j \in \mathcal{N}_i} w_{ij} (x_j^{(l)} - x_i^{(l)}) + \rho_i \sum_{l=0}^{m-1} k_l (x_L^{(l)} - x_i^{(l)}) - k_{m-1} \rho_i u_L. \tag{18}$$

The following result describes the consensus dynamics.

Lemma 4: Using the control law in (18), the dynamics of all the agents, given by (15), evolve along a common consensus dynamics given by

$$x_i^{(m)} + \eta_{m-1} x_i^{(m-1)} + \cdots + \eta_0 x_i = \sum_{l=0}^{m-1} \sum_{j \in \mathcal{N}_i} w_{ij} (x_j^{(l)} - x_i^{(l)})$$

which is independent of the leader.

Proof: See Appendix. ■

Remark 8: From [21, Lemma 4 and Th. 3], it follows that the characteristic equation of the system described by (15) and (18) is given by $\Pi_{j=1}^n p_j(s)$, where $p_j(s) = s^m + (\eta_{m-1} + \lambda_j) s^{m-1} + \cdots + (\eta_0 + \lambda_j)$ and λ_j is the jth eigenvalue of the Laplacian $\mathcal{L}$. Consequently, the system achieves consensus in all the states of the agents 1 to n if the polynomials $p_j(s)$ are Hurwitz for all $j = 2, \ldots, n$. Also, the consensus states evolves along the dynamical system Ξ given by

$$\xi^{(m)} + \eta_{m-1}\xi^{(m-1)} + \cdots + \eta_0 \xi = 0.$$

The following result ensures that consensus among the followers is achieved with bounded states.

Lemma 5: The polynomials in the family $s^m + \sum_{l=0}^{m-1} (\eta_l + [0, \epsilon']) s^l$ belong to the family $\bar{\mathcal{I}}(s,\epsilon)$ for some positive number $\epsilon' < \epsilon$ and are thus Hurwitz.

Proof: See Appendix. ■

Now, we state the main result about the consensus in the states of n heterogeneous linear higher order agents.

Theorem 5: For the multiagent system given by (15), that satisfies Assumption 1, there exist edge weights on the graph $\mathcal{G}$, and the directed edges connecting each agent to the leader, and a reduced order model of the leader, as in (17), to ensure that consensus is achieved in the states of the n followers.

Proof: Remark 8 indicates that for a leader given by (17) and control law (18), the system dynamics depend only on the roots of the equation $\Pi_{j=1}^n p_j(s) = 0$, where $p_j(s) = s^m + (\eta_{m-1} + \lambda_j) s^{m-1} + \cdots + (\eta_0 + \lambda_j)$ and λ_j is the jth eigenvalue of the

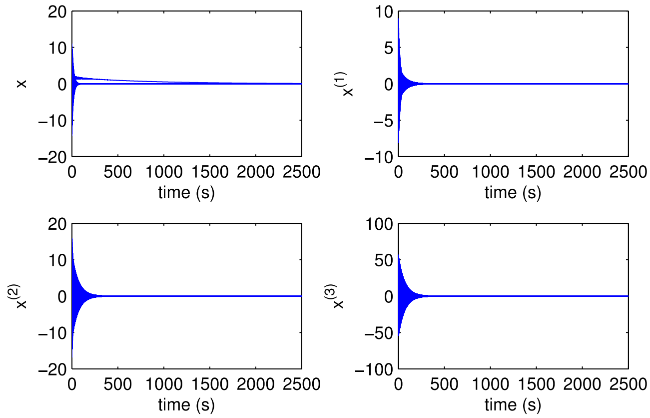

Fig. 3. Consensus in states of 40 fourth-order agents.

Laplacian $\mathcal{L}$. If all the roots are in the open lhp, consensus follows. From Lemma 5 (see Appendix), it follows that there exists a nonempty interval $[0, \epsilon']$ such that every polynomial of the form $s^m + \sum_{l=0}^{m-1}(\eta_l + [0, \epsilon'])s^l$ is Hurwitz. Note that there always exists a choice of edge weights ensuring that $||\mathcal{L}||_2 < \epsilon'$. Hence, $p_j(s) = s^m + (\eta_{m-1} + \lambda_j)s^{m-1} + \cdots + (\eta_0 + \lambda_j)$ belongs to the family of polynomials $s^m + \sum_{l=0}^{m-1}(\eta_l + [0, \epsilon'])s^l$, for a some choice of edge weights and is Hurwitz for all $j = 1, 2, \ldots, n$. Thus, the polynomial $\Pi_{j=1}^{n} p_j(s)$ is also Hurwitz. ■

Remark 9: Note that the dynamics of the leader, given by (17), does not affect the overall consensus dynamics of the agents and thus the leader's dynamics may even be unstable.

Remark 10: Suppose r out of the n agents have identical dynamics described by the set of coefficients $\{\eta_i\}_{i=0,\ldots,m-1}$. Then, the matching condition in (16) needs to hold only for the remaining $n - r$ agents in order to achieve consensus. Moreover, Assumption 1 may also be relaxed if the dynamics of the r identical agents are robustly stable and these r identical agents do not need to be pinned to the leader.

Next, we show that a less stringent condition than Assumption 1 can ensure desired consensus in the followers' states.

Theorem 6: For the multiagent system given by (15), that satisfies (16), there exist edge weights on graph $\mathcal{G}$, and the directed edges connecting each agent to the leader, and a reduced order model of the leader, as in (17), to ensure consensus in the states of the n follower agents, if the characteristic polynomial of at least one follower belongs to the interior of a family of stable interval polynomials.

Proof: In (16), replace η_j with $\alpha_{j,p}$ for all j, where agent p has robustly stable dynamics. Thus, we may choose $\rho_p = 0$, thereby choosing the dynamics of agent p as the consensus dynamics. Now, the proof is similar to that of Theorem 5.

Remark 11: Theorem 6 implies that if Assumption 1 is replaced by a weaker assumption, that at least one of the agents has robustly stable dynamics, consensus with bounded states is achievable.

V. Simulations

Consider Example 2 in Section III-A with 40 agents in S_{40} and $\mu = 0.45$. Fig. 3 shows that all the states of the 40 agents

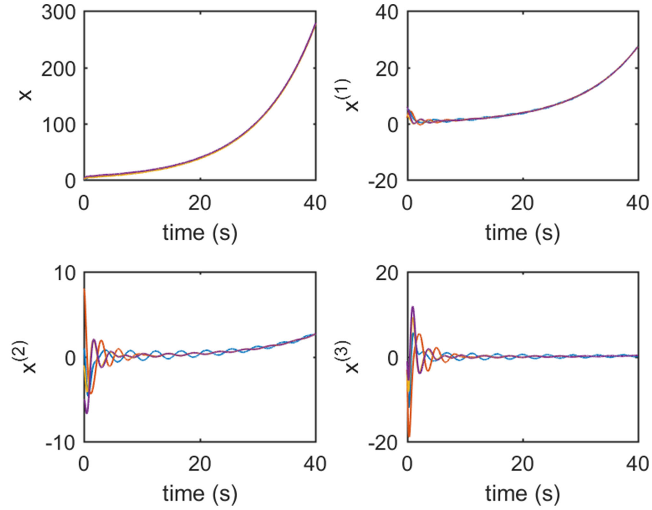

Fig. 4. Consensus in states of fifth-order agents through S_4.

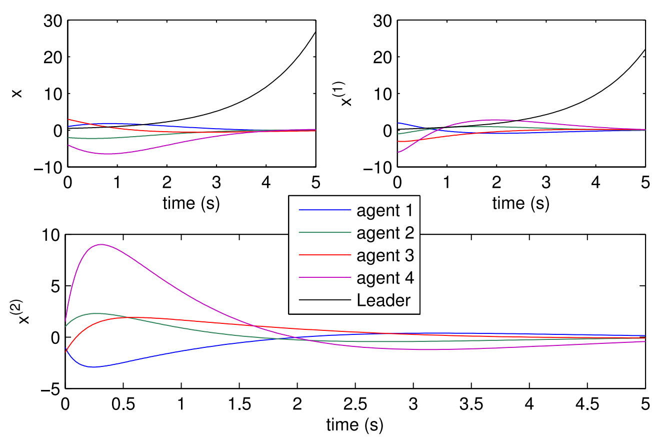

Fig. 5. Consensus among the followers.

converge for the chosen edge weights and given values of $\{\beta_\ell\}_{\ell=0,\ldots,3}$, even though the system does not satisfy the condition of (9). This follows from the discussion about the sufficiency of Theorem 2.

Next, consider a system of inherently unstable identical agents whose linear uncertain dynamics can be described by the characteristic polynomial $s^5 + [16, 20]s^4 + [90, 100]s^3 + [300, 310]s^2 + [340, 350]s + [-40, -35]$ and and $\beta_\ell = 1\, \forall \ell$. Here, Assumption 1 does not hold. Using Kharitonov's Theorem, $\underline{k} = 40$ and $\bar{k} = 192.67$. Thus, the network parameters must be so chosen that $0.2076 < \frac{\lambda_{\min}(E(\mathcal{G}_\tau)^T E(\mathcal{G}_\tau))\lambda_{\min}(RR^T)}{\lambda_{\max}(\mathcal{L}_u)}$. Suppose the network is a star S_n. Now, for $n > 4$, the condition (9) of Theorem 2 is violated for any choice of edge weight μ. Fig. 4 shows the consensus in the first four states for four agents with $\mu = 42$. The states are, however, not bounded at consensus.

Finally, consider a system of four heterogeneous third-order agents over C_4 with dynamics given by

$$\dddot{x}_i + \alpha_{2i}\ddot{x}_1 + \alpha_{1i}\dot{x}_i + \alpha_{0i}x_i = u_i,\ i = 1, 2, 3, 4$$

with $(\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}) = (8, 7, 5, 11)$, $(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}) = (10, 9, 7, 13)$, and $(\alpha_{01}, \alpha_{02}, \alpha_{03}, \alpha_{04}) = (3.5, 5, 8, -3)$. The matching condition in (16) is satisfied and with a leader given by $\ddot{x}_L + \dot{x}_L - 1.5x_L = u_L$, $\rho_1 = -1$, $\rho_2 = 0$, $\rho_3 = 2$, $\rho_4 = -4$, consensus is achieved among the four agents, as seen in Fig. 5, despite one of these agents having unstable open-loop dynamics (violation of Assumption 1). The external input to the leader is zero. Though the leader's dynamics are unstable, consensus among the pinned agents is unaffected.

VI. Conclusion

Using Kharitonov's Theorem as a tool for both design and analysis, this paper provided a relationship between the uncertainty bounds characterized by gain margins and the parameters of the network. Further, the amount of edge weight perturbations tolerable by such an uncertain system of agents was also obtained. Moreover, without using a separate local controller for each higher order agent, a pinning control structure was proposed for a system of heterogeneous agents to achieve consensus, by designing the dynamics of the leader along with suitable edge weights in the network.

In future, uncertain nonlinear systems may be considered in a similar setup, to relate stability margins of agent models with properties of the network. Also, extension of the present analysis to directed networks will provide an interesting new direction of investigation.

Appendix

Proof of Lemma 1

Proof: Comparing (6) and (7), it is apparent that if the gain k in Fig. 2 is an eigenvalue of $\mathcal{L}$, then the closed-loop poles of (7) are zeros of the polynomial (6). Now, if all the nonzero eigenvalues of $\mathcal{L}$ belong to the interval $(\underline{k}, \bar{k})$, then the zeros of $P(s)$ in (6), except the roots of the agent's characteristic equation (that is, all the zeros of $\bar{P}(s)$), will be in the lhp. Application of [21, Th. 3] proves consensus. ■

Proof of Lemma 3

Proof: For any agent i, choose ρ_i such that $\max_j |k_j \rho_i| < \epsilon$. Subsequently, setting the ratio in (16) as ρ_i, one obtains the set $\{\eta_0, \eta_1, \ldots, \eta_{m-1}\}$. This choice of $\{\eta_0, \eta_1, \ldots, \eta_{m-1}\}$ satisfies the condition (16) for any other agent p with a corresponding ρ_p. Moreover, due to Assumption 1, the characteristic equation of the system Ξ described by $\xi^{(m)} + \eta_{m-1}\xi^{(m-1)} + \cdots + \eta_0\xi = 0$ belongs to $\bar{\mathcal{I}}(s, \epsilon)$, since $|\eta_j - \alpha_{j,i}| < \epsilon \forall j$ for some $\epsilon > 0$, and the characteristic polynomial of agent i belongs to $\bar{\mathcal{I}}(s, 0)$. Thus, the plant Ξ is Hurwitz. ■

Proof of Lemma 4

Proof: Using the control law (18) in (15), the dynamics of agent i transforms to

$$x_i^{(m)} + \eta_{m-1}x_i^{(m-1)} + \cdots + \eta_0 x_i = \sum_{l=0}^{m-1} \sum_{j \in \mathcal{N}_i} w_{ij}(x_j^{(l)} - x_i^{(l)}). \tag{19}$$

The coefficients η_j, $j = 0, 1, \ldots, m-1$ are independent of i and the dynamics of the leader do not appear in (19). The inter-agent dynamics on the right-hand side of (19) can be described by the Laplacian. Thus, the dynamics of the system (15), under the control law (18), is given by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & \boldsymbol{I}_n & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{I}_n \\ -\eta_0\boldsymbol{I}_n - \mathcal{L} & -\eta_1\boldsymbol{I}_n - \mathcal{L} & \cdots & -\eta_{m-1}\boldsymbol{I}_n - \mathcal{L} \end{bmatrix}. \tag{20}$$

We note that (20) is independent of the leader's dynamics.

Proof of Lemma 5

Proof: The coefficients of the monic polynomial $s^m + \sum_{l=0}^{m-1} \eta_l s^l$ satisfy $\underline{\alpha}_l - \epsilon < \eta_l < \bar{\alpha}_l + \epsilon \ \forall l$ (from the proof of Lemma 3). It then follows that there exists some $\epsilon' < \epsilon$ such that $\underline{\alpha}_l - \epsilon < \eta_l + \epsilon' < \bar{\alpha}_l + \epsilon \ \ \forall l$. Moreover, for any $0 < \delta < \epsilon'$, the inequality $\underline{\alpha}_l - \epsilon < \eta_l + \delta < \bar{\alpha}_l + \epsilon \ \ \forall l$ holds. This implies that any polynomial in the family $s^m + \sum_{l=0}^{m-1}(\eta_l + [0, \epsilon'])s^l$ also belongs to the family of interval polynomials, $\bar{\mathcal{I}}(s, \epsilon)$, for some $\epsilon > \epsilon' > 0$, and is thus Hurwitz. ■

Lemma 6: For a connected undirected graph having n nodes and less than $2n - 2$ distinct edges, $\sigma_{\min}(RR^T) = 1$.

Proof: Since $T_\tau \in \mathbb{R}^{(n-1)\times(|\mathcal{E}|-n+1)}$, if $|\mathcal{E}| < 2n - 2$, we have $\lambda_{\min}(T_\tau T_\tau^T) = 0$. Now, $\sigma_{\min}(RR^T) = \lambda_{\min}(I_{n-1} + T_\tau T_\tau^T) = 1 + \lambda_{\min}(T_\tau T_\tau^T) = 1 + 0$. ■

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