4th IFAC Workshop on Distributed Estimation
and Control in Networked Systems
September 25-26, 2013. Rhine-Moselle-Hall, Koblenz, Germany

On the Steady-State Inverse-Optimality of Passivity-based Cooperative Control

Mathias Bürger [*,1] **Daniel Zelazo** [**,1] **Frank Allgöwer** [*,1]

[*] *Institute for Systems Theory and Automatic Control, University of Stuttgart, Pfaffenwaldring 9, 70550 Stuttgart, Germany (e-mail: {buerger, allgower}@ ist.uni-stuttgart.de).*
[**] *Faculty of Aerospace Engineering, Technion - Israel Institute of Technology, Haifa 32000, Israel (e-mail: dzelazo@technion.ac.il).*

Abstract: We consider a passivity based cooperative control problem, and show that the steady-state behavior of the networked system is intimately related to a family convex network optimization problems. This result provides a duality interpretation between the different signals in the cooperative control system. In particular, we show that the input and output signals as well as the dynamic controller state and controller output are pairs of dual variables. The presented results facilitate an optimal controller design in networked systems. We show how this novel interpretation leads the way to an optimal routing design in distribution networks.

1. INTRODUCTION

A recent trend in modern control theory is the study of cooperative control problems amongst groups of dynamical systems that interact over an information exchange network. The fundamental goal for the analysis of these systems is to reveal the interplay between properties of the individual dynamic agents, the underlying network topology, and the interaction protocols that influence the functionality of the overall system. Amongst the numerous control theoretic approaches being pursued to define a general theory for networks of dynamical systems, *passivity* takes an outstanding role; see e.g., Bai et al. [2011]. In Arcak [2007], a passivity based framework for group coordination problems was established. Passivity was used in Zelazo and Mesbahi [2010] to derive performance bounds on the input/output behavior of consensus-type networks. Passivity is also widely used in coordinated control of robotic systems (Chopra and Spong [2006]) or for cooperative control with quantized measurements (De Persis and Jayawardhana [2012]). The refined concept of incremental passivity provides a framework to study various synchronization problems (Stan and Sepulchre [2007], Scardovi et al. [2010]). Passivity was also used in the context of Port-Hamiltonian systems on graphs, to establish a unifying framework for a variety of networked dynamical system in van der Schaft and Maschke [2012]. In a previous work, we used a passivity-like framework to study *clustering* in networks of heterogeneous scalar dynamical systems with saturated couplings (Bürger et al. [2013, 2011]).

We consider in this paper a canonical passivity-based cooperative control framework comprised of equilibrium independent passive systems (see Hines et al. [2011]) and dynamic controllers interconnected by the networked structure. We show that, even without specifying the dynamic controllers, the output agreement steady-state input and output of the plants can be understood as a primal/dual pair of variables associated to a network optimization problem. Furthermore, we propose an internal model based control law for the output agreement problem and show that the controller state and the controller output can again be connected to a primal/dual solution of a network optimization problem. By exploiting these duality relations, it becomes straight-forward to construct a Lyapunov function to prove convergence of the closed-loop system. We then proceed to examine a distribution problem in inventory system and show how this novel analysis methods lead directly to a distributed controller design that solves the optimal routing problems.

The remainder of the paper is organized as follows. The problem set-up is introduced in Section 2. The inverse optimality of the feasible agreement steady-state in an open-loop configuration is discussed in Section 3. The inverse optimality of the controller states and the controller outputs is discussed in Section 4, before a stability analysis of the closed-loop system is presented in 5. The implications of the results for controller design is presented in Section 6 with an example related to dynamic distribution networks. Finally, concluding remarks are offered in Section 7.

Preliminaries: A *graph* $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, consists of a finite set of *nodes* and *edges*, denoted as $\mathbf{V} = \{v_1, \dots, v_{|\mathbf{V}|}\}$ and $\mathbf{E} = \{e_1, \dots, e_{|\mathbf{E}|}\}$, respectively. The notation $e_k = (v_i, v_j) \in \mathbf{E} \subset \mathbf{V} \times \mathbf{V}$ indicates that v_i is the initial node of edge e_k and v_j is the terminal node, abbreviated as $k = (i, j)$, and written as $k \in \mathbf{E}$ and $i, j \in \mathbf{V}$. [2] The *incidence matrix* $E \in \mathbb{R}^{|\mathbf{V}|\times|\mathbf{E}|}$ of the graph $\mathcal{G}$ with arbitrary orientation, is a $\{0, \pm 1\}$ matrix with the rows and columns indexed by the nodes and edges of $\mathcal{G}$ such that $[E]_{ik}$ has value '+1' if node i is the initial node of edge k, '-1' if it is the terminal node, and '0' otherwise. This definition implies that for any graph, $\mathbf{1}^\top E = 0$, where $\mathbf{1} \in \mathbb{R}^{|\mathbf{V}|}$ is the vector of all ones. We refer sometimes to the *agreement space* of $\mathcal{G}$ as the null space $\mathcal{N}(E^\top)$. Additionally, $\mathcal{N}(E)$ is named the *circulation space* of $\mathcal{G}$, and $\mathcal{R}(E^\top)$ the *differential space*, see Godsil and Royle [2001].

[1] The authors thank the German Research Foundation (DFG) for financial support of the project within the Cluster of Excellence in Simulation Technology (EXC 310/2) at the University of Stuttgart.

[2] Although we consider undirected graphs, we assign arbitrary orientations to each edge

2. PROBLEM STATEMENT

We study a canonical model for passivity-based cooperative control. Consider a network $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ of dynamical systems, with each node of $\mathcal{G}$ representing a single-input/single-output nonlinear dynamical system,

$$\Sigma_i: \quad \begin{aligned} \dot{x}_i(t) &= f_i(x_i(t), u_i(t)), \\ y_i(t) &= h_i(x_i(t), u_i(t)), \qquad i \in \mathbf{V}, \end{aligned} \tag{1}$$

with state $x_i(t) \in \mathbb{R}^{n_i}$, control input $u_i(t) \in \mathbb{R}$ and output $y_i(t) \in \mathbb{R}$. The standing assumption we impose on the systems throughout paper is *output strictly equilibrium independent passivity* (OSEIP), see Hines et al. [2011] to which we refer the reader for a detailed discussion on OSEIP. This property can be summarized as follows

Assumption 2.1. (OSEIP). There exists a set $\mathcal{U}_i \subseteq \mathbb{R}$ and a single-valued map $k_{\mathrm{y},i} : \mathcal{U}_i \mapsto \mathbb{R}$, such that for all $\mathrm{u}_i \in \mathcal{U}_i$ there exists a positive definite C^1 function $S_{\mathrm{u}}(x_i)$ satisfying

$$\dot{S}_{\mathrm{u}}(x_i(t), u_i(t)) \leq -\gamma_i \|y_i(t) - \mathrm{y}_i\|^2 + (u_i(t) - \mathrm{u}_i)(y_i(t) - \mathrm{y}_i), \tag{2}$$

with $\mathrm{y}_i = k_{\mathrm{y},i}(\mathrm{u}_i)$, and $u_i(t)$ and $y_i(t)$ corresponding to the input and output of the system in (1) respectively.

The map $k_{\mathrm{y},i}$ is called the *equilibrium input-to-output map* of system (1). It characterizes the equilibrium output of the control system under a constant input signal from $\mathcal{U}_i$. The existence of a storage function (2) implies that the map $k_{\mathrm{y},i}$ is co-coercive, see Hines et al. [2011]. We impose here a slightly stronger assumption.

Assumption 2.2. For all $i \in \{1, \ldots, |V|\}$, it holds that $\mathcal{U}_i = \mathbb{R}$, $\mathcal{Y}_i = \mathbb{R}$ and the equilibrium input-to-output maps $k_{\mathrm{y},i}(\mathrm{u}_i)$ are invertible and strongly monotone.

Two important system classes, satisfying the previous assumptions are the following.

Example 2.3. (Affine Systems). Consider the affine system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \mathrm{w} \\ y(t) &= Cx(t) + Du(t) + \mathrm{v}, \end{aligned} \tag{3}$$

with (A, B) controllable and (A, C) observable. The system is OSEIP if (A, B, C, D) satisfy the matrix equations of the KYP-lemma (Khalil [2002]), (i.e., the system is strictly output passive in the classical sense) and A is invertible (Hines et al. [2011]). In addition, with $\mathrm{w} \in \mathbb{R}^p$ and $\mathrm{v} \in \mathbb{R}$ being constant signals the equilibrium input-output map is the affine function

$$k_y(\mathrm{u}) = \left(-CA^{-1}B + D\right)\mathrm{u} + \left(-CA^{-1}\mathrm{w} + \mathrm{v}\right), \tag{4}$$

i.e., the *dc-gain* of the linear system plus the constant value determined by the exogenous inputs.

Example 2.4. (Gradient Systems). Consider the scalar nonlinear system

$$\dot{x}(t) = -f(x(t)) + u(t), \; y(t) = x(t), \tag{5}$$

with $x(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$, and $y(t) \in \mathbb{R}$. The system is OSEIP if (5) satisfies the QUAD condition, see DeLellis et al. [2011], i.e., for all $x'(t)$ and $x''(t)$ it holds that

$$(x'(t) - x''(t))(f(x'(t)) - f(x''(t))) \geq \gamma(x'(t) - x''(t))^2.$$

Systems of this form are considered in the context of synchronization (Scardovi et al. [2010], DeLellis et al. [2011]). Note that the QUAD condition reduces for the scalar system (5) to a strong monotonicity condition on $f(x(t))$. The dynamics can then be understood as the gradient of a strongly convex function, i.e., $f(x(t)) = \nabla F(x(t))$. This corresponds to the system class studied in Bürger et al. [2010, 2011, 2013]. The equilibrium input-output map is then the strongly monotone function $k_{\mathrm{y}}(\mathrm{u}) = f^{-1}(\mathrm{u})$.

To avoid trivial cases, we implicitly assume throughout the paper that the systems are heterogeneous. In particular, we assume that the systems without an external forcing would all have different equilibrium outputs, i.e., $k_{\mathrm{y},i}(0) \neq k_{\mathrm{y},j}(0)$.

In the following we will adopt the notation $\boldsymbol{y}(t) = [y_1(t), \ldots, y_{|\mathbf{V}|}(t)]^\top$ and $\boldsymbol{u}(t) = [u_1(t), \ldots, u_{|\mathbf{V}|}(t)]^\top$ for the stacked output and input vectors of the complete network. We use normal bold font letters $\mathbf{y}, \mathbf{u}$ to indicate that a vector corresponds to equilibrium trajectories. Similarly, we use $\mathbf{k}_{\mathrm{y}}(\mathbf{u}) = [k_{\mathrm{y},1}(\mathrm{u}_1), \ldots, k_{\mathrm{y},|\mathbf{V}|}(\mathrm{u}_{|\mathbf{V}|})]^\top$. If all $k_{\mathrm{y},i}$ are invertible, we write $\mathbf{k}_{\mathrm{y}}^{-1}(\mathbf{y}) = [k_{\mathrm{y},1}^{-1}(\mathrm{y}_1), \ldots, k_{\mathrm{y},n}^{-1}(\mathrm{y}_{|\mathbf{V}|})]^\top$.

The control objective we consider here is output agreement on a constant steady-state value.

Definition 2.5. A network of dynamical systems (1) is said to reach *output agreement* if

$$\lim_{t\to\infty} \boldsymbol{y}(t) \to \beta \mathbf{1} \tag{6}$$

for some $\beta \in \mathbb{R}$, called the *agreement value*.

Output agreement should be achieved through a coupling of the network nodes using the control inputs. The cooperative control framework we consider is based on a canonical control structure, illustrated in Figure 1.

The controllers Π_k are dynamical systems located on the edges of the network,

$$\Pi_k: \quad \begin{aligned} \dot{\eta}_k(t) &= g_k(\eta_k(t), \zeta_k(t)) \\ \mu_k(t) &= \psi_k(\eta_k(t), \zeta_k(t)), \qquad k \in \mathbf{E}, \end{aligned} \tag{7}$$

where $\eta_k(t) \in \mathbb{R}^{q_k}$ is the internal state of the controller and $\zeta_k(t) \in \mathbb{R}$ is its input. In fact, the controller inputs are precisely the *relative outputs* of the systems Σ_i. In a stacked vector form the controller inputs take the form

$$\boldsymbol{\zeta}(t) = E^\top \boldsymbol{y}(t). \tag{8}$$

As we consider the underlying graph to be *undirected*, we focus on symmetric couplings, where the output of a controller influences the two incident systems with reversed signs. Following the network interpretation, the control input is generated by the mapping of the controller outputs, i.e.

$$\boldsymbol{u}(t) = -E\boldsymbol{\mu}(t). \tag{9}$$

This structure has evolved as a standard control structure in passivity-based cooperative control, see e.g. Bai et al. [2011], Arcak [2007], van der Schaft and Maschke [2012], and is the basis for our work here. Output agreement of such networks under time-varying external signals is studied in Bürger and De Persis [2013]. We investigate here the optimality properties of the steady states appearing in this cooperative control framework.

3. OPTIMALITY OF THE STEADY-STATE OPEN-LOOP OUTPUT AGREEMENT

We make the following observation resulting from the network structure of the control system. The control input $\boldsymbol{u}(t)$ satisfies by construction

$$\boldsymbol{u}(t) \in \mathcal{R}(E). \tag{10}$$

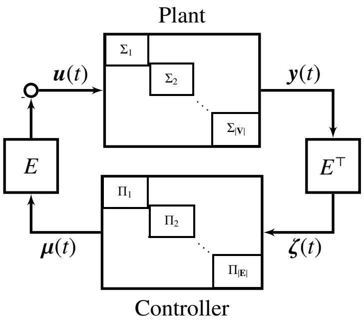

Fig. 1. A canonical cooperative control structure.

This structural condition is independent of the exact controller we choose, but is purely a consequence of the networked structure of Figure 1. We first investigate the role of the structural constraint $\boldsymbol{u}(t) \in \mathcal{R}(E)$ in the agreement problem.

Definition 3.1. A pair $(\mathbf{u}, \mathbf{y})$ is said to be an *output agreement steady-state input-output pair* if $\mathbf{u} \in \mathcal{R}(E)$, $\mathbf{y} \in \mathcal{N}(E^\top)$ and $\mathbf{y} = \mathbf{k}_{\mathrm{y}}(\mathbf{u})$.

The first algebraic constraint $\mathbf{u} \in \mathcal{R}(E)$ contributes to the structure of the coupling controller. The second constraint $\mathbf{y} \in \mathcal{N}(E^\top)$ enforces the output agreement since $\mathcal{N}(E^\top) = \mathrm{span}\{\mathbf{1}\}$. Finally, the condition $\mathbf{y} = \mathbf{k}_{\mathrm{y}}(\mathbf{u})$ ensures that $\mathbf{u}$ and $\mathbf{y}$ are an equilibrium input output pair.

We connect the output-agreement steady-state now to a dual pair of optimization problems, namely an *optimal potential* and an *optimal flow* problem, respectively. The two problems take the standard form of network optimization problems as defined in Rockafellar [1998]. We define for each node the *integral function* of the equilibrium input-to-output map $k_{\mathrm{y},i}(\mathrm{u}_i)$, denoted $K_i(\mathrm{u}_i)$, and satisfying

$$\nabla_{\mathrm{u}_i} K_i(\mathrm{u}_i) = k_{\mathrm{y},i}(\mathrm{u}_i). \tag{11}$$

From Assumption 2.2, $k_{\mathrm{y},i}$ is strongly monotone, and thus the integral function K_i is convex. We will call $K_i(\mathrm{u}_i)$ in the following the *cost function* of node i. The convex conjugate of the cost function $K_i(\mathrm{u}_i)$ is

$$K_i^\star(\mathrm{y}_i) = \sup_{\mathrm{u}_i} \{\mathrm{y}_i \mathrm{u}_i - K_i(\mathrm{u}_i)\}, \tag{12}$$

and will be called in the following the *potential function* of node i. Recall that $\nabla_{\mathrm{y}} K_i^\star(\mathrm{y}_i) = k_{\mathrm{y},i}^{-1}(\mathrm{y}_i)$. Consider now the following dual pair of network optimization problems.

Optimal Potential Problem: Consider the static *optimal potential problem* of the form

$$\min_{\mathrm{y}_i} \sum_{i=1}^{|\mathbf{V}|} K_i^\star(\mathrm{y}_i), \qquad \text{s.t.} \quad \mathbf{y} \in \mathcal{N}(E^\top). \tag{OPP1}$$

The objective functions of this problem are the convex conjugates of the integral functions of the equilibrium input-to-output maps. The constraint $\mathbf{y} \in \mathcal{N}(E^\top)$ (i.e., $E^\top \mathbf{y} = 0$) enforces a balancing of the variables, i.e., $\mathrm{y}_1 = \cdots = \mathrm{y}_{|\mathbf{V}|}$. We will call $\mathbf{y}$ in the following potential variables.

Optimal Flow Problem: The dual problem to (OPP1) is the *optimal flow problem*

$$\min_{\mathrm{u}_i} \sum_{i=1}^{|\mathbf{V}|} K_i(\mathrm{u}_i) \qquad \text{s.t.} \quad \mathbf{u} \in \mathcal{R}(E). \tag{OFP1}$$

The vector $\mathbf{u} \in \mathbb{R}^{|\mathbf{V}|}$ will be called in the following the *divergence* of the network.

We can now connect the dual pair of network optimization problems to the output agreement steady-state.

Theorem 3.2. Consider the network $\mathcal{G}$ with node dynamics (1) and networked input-output channels (8), (9), then the following statements are equivalent:

(i) $(\mathbf{u}, \mathbf{y})$ are are the steady-state output agreement pairs for the system input and output, respectively;
(ii) $(\mathbf{y}, \mathbf{u})$ is a primal/dual solution pair to (OPP1);
(iii) $(\mathbf{u}, \mathbf{y})$ is a primal/dual solution pair to (OFP1).

Proof. An output agreement steady-state input-output pair is such that $\mathbf{y} = \mathbf{k}_{\mathrm{y}}(\mathbf{u}) = \beta\mathbf{1}$, for some $\mathbf{u} \in \mathcal{R}(E)$. By assumption, all input-output characteristics are invertible, such that $\mathbf{u} = \mathbf{k}_{\mathrm{y}}^{-1}(\beta\mathbf{1})$. From $\mathbf{u} \in \mathcal{R}(E)$, it follows that $\mathbf{1}^\top\mathbf{u} = \mathbf{1}^\top\mathbf{k}_{\mathrm{y}}^{-1}(\beta\mathbf{1}) = 0$. To see now the equivalence to (ii), consider the Lagrangian of (OPP1)

$$\mathcal{L}(\mathbf{y}, \mathbf{u}) = \sum_{i=1}^{|\mathbf{V}|} K_i^\star(\mathrm{y}_i) - \mathbf{u}^\top\mathbf{y}, \quad \mathbf{u} \in \mathcal{R}(E).\ ^3$$

The optimal primal dual solution pair is such that $\mathbf{y}$ is a minimizer of $r(\mathbf{y}) = \max_{\mathbf{u}\in\mathcal{R}(E)} \mathcal{L}(\mathbf{y}, \mathbf{u})$, and $\mathbf{u} \in \mathcal{R}(E)$ is a maximizer of $s(\mathbf{u}) = \min_{\mathbf{y}} \mathcal{L}(\mathbf{y}, \mathbf{u})$. Now, note that $r(\mathbf{y})$ takes the finite value $\sum_{i=1}^{|\mathbf{V}|} K_i^\star(\mathrm{y}_i)$ for all $\mathbf{y} \in \mathrm{span}\{\mathbf{1}\}$ (i.e., $\mathbf{u}^\top\mathbf{y} = 0$) and is unbounded otherwise. Thus, we have $\mathbf{y} = \beta\mathbf{1}$. The optimality condition for $r(\mathbf{y})$ reduces now to

$$\sum_{i=1}^{|\mathbf{V}|} \nabla K_i^\star(\beta) = \sum_{i=1}^{|\mathbf{V}|} k_{\mathrm{y},i}^{-1}(\beta) = 0,$$

which is exactly the condition for an output agreement steady-state. Additionally, from the optimality condition $\nabla_{\mathbf{y}}\mathcal{L}(\mathbf{y}, \mathbf{u}) = 0$ follows that $\mathbf{u} = \mathbf{k}_{\mathrm{y}}^{-1}(\mathbf{y})$. Thus, the primal dual solution to (OPP1) is an output agreement steady-state input-output pair. Finally, to see (iii) it sufficies to see that (OFP1) is exaclty identical to $\min_{\mathbf{u}\in\mathcal{R}(E)}(-s(\mathbf{u}))$, where $s(\mathbf{u}) = \min_{\mathbf{y}} \mathcal{L}(\mathbf{y}, \mathbf{u})$.

This result shows that the equilibrium input output pair in a network of equilibrium independent passive systems is intimately connected to a pair of dual network optimization problems, i.e. (OPP1) and (OFP1). We can now conclude that the equilibrium input output pair has an inverse optimality property, since the optimization problem solved by this steady-state configuration is defined only by the systems dynamics and the network topology.

4. OPTIMALITY OF THE CONTROLLER STEADY-STATE

We now close the control loop and present a distributed control scheme that ensures convergence to the agreement steady-state. We focus on controls in the generic structure illustrated in Figure 1 and define the dynamical systems Π_k. To begin, note that any static diffusive coupling, using only the current relative measurement, i.e., $(y_i(t) - y_j(t))$, vanishes as an agreement state is reached. Consequently, for systems with different unforced equilibria, output synchronization cannot be achieved using only static couplings. As a direct consequence of the *internal model principle for synchronization*, see Wieland et al.

[3] Note that $\mathbf{u}$ is not the true Lagrange multiplier for the problem, but is interpreted as such by observing it satisfies $\mathbf{u} = E\lambda$ for the true multiplier λ.

[2011], we consider dynamic couplings and use an integrator as common internal model for the entire network of the form

$$\Pi_k: \quad \begin{aligned} \dot{\eta}_k(t) &= \zeta_k(t) \\ \mu_k(t) &= \psi_k(\eta_k(t)). \end{aligned} \tag{13}$$

We refer to Bürger and De Persis [2013] for a detailed explanation on how such a controller can be derived from the internal model principle. The function $\psi_k : \mathbb{R} \mapsto \mathbb{R}$ is a coupling nonlinearity which, similar to Arcak [2007], is assumed to be the gradient of some convex, differentiable function $P_k : \mathbb{R} \mapsto \mathbb{R}_{\geq 0}$, attaining a minimum at the origin, i.e.,

$$\psi_k(\eta_k(t)) := \nabla P_k(\eta_k(t)). \tag{14}$$

The control input applied to the plants is computed according to the structure of Figure 1 and is thus given by (9), i.e. $\boldsymbol{u}(t) = -E\boldsymbol{\mu}(t)$. Summarizing, the dynamic coupling control law can be represented as

$$\begin{aligned} \dot{\boldsymbol{\eta}}(t) &= E^\top \boldsymbol{y}(t), \quad \boldsymbol{\eta}(t_0) \in \mathcal{R}(E^\top), \\ \boldsymbol{u}(t) &= -E\boldsymbol{\psi}(\boldsymbol{\eta}(t)), \end{aligned} \tag{15}$$

where $\boldsymbol{\eta}(t) = [\eta_1(t), \dots, \eta_{|\mathbf{E}|}(t)]^\top$ is the controller state and the controller output is denoted as $\boldsymbol{\psi}(\boldsymbol{\eta}(t))$. Please note that $\boldsymbol{\eta}(t) \in \mathcal{R}(E^\top)$ and $\boldsymbol{u}(t) \in \mathcal{R}(E)$ for all times by construction. The coupling nonlinearities ψ_k are a degree of freedom in the control design. However, in this paper, we impose the following requirement.

Assumption 4.1. For all $k \in \{1, \dots, |\mathbf{E}|\}$, the functions $P_k(\eta_k(t))$ are twice differentiable, even, and strongly convex on $\mathbb{R}$.

The coupling controller (15) is only a suitable controller for the output agreement problem if it is able to generate the steady-state control input $\mathbf{u}$ described by Theorem 3.2.

Definition 4.2. A vector $\upeta \in \mathbb{R}^{|\mathbf{E}|}$ is called an *output-agreement controller steady-state* if $\mathbf{u} = -E\boldsymbol{\psi}(\upeta)$, with $\mathbf{u}$ being a solution to (OFP1). Correspondingly, the vector $\upmu = \boldsymbol{\psi}(\upeta)$ is called the *output-agreement controller output*.

We now connect the steady-state behavior of the controller again to a dual pair of network optimization problems.

Optimal Potential Problem: We assume in the following that $\mathbf{u} = [\mathrm{u}_1, \dots, \mathrm{u}_{|\mathbf{V}|}]^\top$ is a solution to (OFP1). Consider now the following optimization problem

$$\begin{aligned} \min_{\upeta, \mathbf{v}} \quad & \sum_{i=1}^{|\mathbf{V}|} \mathrm{u}_i \mathrm{v}_i + \sum_{k=1}^{|\mathbf{E}|} P_k(\upeta_k), \\ \text{s.t.} \quad & \upeta = E^\top \mathbf{v}. \end{aligned} \tag{OPP2}$$

Please note that although (OPP2) is an optimal potential problem, it is not directly connected to the problem (OPP1), which was associated to the outputs of the dynamical system. Instead, (OPP2) is an additional optimal potential problem, which, as we will show later on, describes the steady-state behavior of the internal model controller (15).

Optimal Flow Problem: Let again $\mathbf{u}$ be a solution to (OFP1). Consider the following optimal flow problem

$$\begin{aligned} \min_{\upmu} \quad & \sum_{k=1}^{|\mathbf{E}|} P_k^\star(\upmu_k) \\ \text{s.t.} \quad & \mathbf{u} + E\upmu = 0. \end{aligned} \tag{OFP2}$$

Please note that in this problem, $\mathbf{u}$ is not a decision variable, but the solution previously defined by (OFP1).

To complete the connection between the dynamic variables of the control systems and the network theory variables, we now formalize the connection between the optimal flow and the output of the internal model controller.

Theorem 4.3. Consider the controller (15) and let $\mathbf{u}$ be a output-agreement steady-state input. Then, the following statements are equivalent:

(i) $(\upeta, \upmu)$ are the steady-state output agreement pairs for the controller state and output, respectively;
(ii) $(\upeta, \upmu)$ is a primal/dual solution pair to (OPP2);
(iii) $(\upmu, \upeta)$ is a primal/dual solution pair to (OFP2).

Proof. The output agreement controller steady state is characterized by the relation $\mathbf{u} + E\nabla P(\upeta) = 0$. The corresponding output agreement contoller output is then $\upmu = \nabla P(\upeta)$. Consider now the problem (OPP2). After replacing the variable $\upeta$ in the objective function with $E^\top \mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^{|\mathbf{V}|}$, it can be easily verified that the first order optimality condition is $\mathbf{u} + E\nabla P(E^T v) = 0$. Thus, the primal solution $\upeta = E^T \mathbf{v}$ is the output agreement controller steady state. Furthermore, consider the Lagrangian of (OPP2) with multiplier $\upmu$, i.e.,

$$\mathcal{L}(\upeta, \mathbf{v}, \upmu) = \mathbf{u}^\top \mathbf{v} + \mathbf{P}(\upeta) + \upmu^\top(-\upeta + E^\top \mathbf{v}).$$

The optimality conditions (KKT-conditions) can be derived from the Lagrangian as

$$\mathbf{u} + E\upmu = 0, \quad \nabla \mathbf{P}(\upeta) - \upmu = 0, \quad -\upeta + E^\top \mathbf{v} = 0.$$

It follows directly that if $\upeta$ is an optimal primal solution then $\upmu = \nabla \mathbf{P}(\upeta)$ is an optimal dual solution. This proves the equivalence of (i) and (ii). Finally, the equivalence of (iii) follows directly after noting that (OFP2) is equivalent to the dual optimization problem $\min_{\upmu}(-s(\upmu))$, where $s(\upmu) = \min_{\upeta, \mathbf{v}} \mathcal{L}(\upeta, \mathbf{v}, \upmu)$.

We have now a second duality relation in the passivity-based cooperative control framework. We want to emphasize that we identified network optimization problems on two different levels of the cooperative control problem. First, at the plant level, the dual problems (OPP1) and (OFP1) characterize the properties of the agreement state. The problems are fully determined by the properties of the dynamical systems (1) and the topology of the network $\mathcal{G}$. Second, another dual pair of network optimization problems, i.e., (OPP2) and (OFP2), is associated to the internal model controller (15), used to achieve output synchronization. These problems depend on the chosen control structure, i.e., the coupling nonlinearities, as well as on the solution to (OFP1). Thus, the two levels are not completely independent, but the plant level problems influence the problems on the control level.

5. STABILITY ANALYSIS

It remains to analyze the behavior of the closed-loop dynamical system. We had two important relations between the signals $\boldsymbol{\zeta}(t) = E^\top \boldsymbol{y}(t)$ and $\boldsymbol{u}(t) = -E\boldsymbol{\mu}(t)$, i.e., the interconnection conditions (8), (9). The two formulas can be combined to what is known as the *conversion formula*, see Rockafellar [1998]:

$$\boldsymbol{\mu}^\top(t)\boldsymbol{\zeta}(t) = -\boldsymbol{y}^\top(t)\boldsymbol{u}(t). \tag{16}$$

The right hand side of this dynamic conversion formula is reminiscent of a supply function for passive dynamical systems. A natural question to ask in the context of this paper is what the conversion formula looks like for the supply function $(\boldsymbol{y}(t) - \mathbf{y})^\top(\boldsymbol{u}(t) - \mathbf{u})$. Exploiting the previously established connections, we make the following considerations

$$\begin{aligned}(\boldsymbol{y}(t)-\mathbf{y})^\top(\boldsymbol{u}(t)-\mathbf{u}) &= -\boldsymbol{y}^\top(t)E(\boldsymbol{\mu}(t)-\mathrm{\mu})-\mathbf{y}^\top(\boldsymbol{u}(t)-\mathbf{u})\\ &= -\boldsymbol{\zeta}^\top(t)(\boldsymbol{\mu}(t)-\mathrm{\mu})-\mathbf{y}^\top(\boldsymbol{u}(t)-\mathbf{u}) \qquad (17)\\ &= -\dot{\boldsymbol{\eta}}^\top(t)\Big(\nabla\mathbf{P}(\boldsymbol{\eta}(t))-\nabla\mathbf{P}(\mathrm{\eta})\Big)-\mathbf{y}^\top(\boldsymbol{u}(t)-\mathbf{u}).\end{aligned}$$

Observe that in this particular problem we have $\mathbf{y}\in\mathrm{span}\{\mathbf{1}\}$ and $\boldsymbol{u}(t),\mathbf{u}\in\mathcal{R}(E)$. Therefore, $\mathbf{y}^\top(\boldsymbol{u}(t)-\mathbf{u})=0$. We conclude that

$$\Big(\nabla\mathbf{P}(\boldsymbol{\eta}(t))-\nabla\mathbf{P}(\mathrm{\eta})\Big)^\top\dot{\boldsymbol{\eta}}(t)=-(\boldsymbol{y}(t)-\mathbf{y})^\top(\boldsymbol{u}(t)-\mathbf{u}). \qquad (18)$$

The last equation has the flavor of a dissipation equality. In fact, a storage function corresponding to (18) is

$$\mathbf{B}_P(\boldsymbol{\eta}(t),\mathrm{\eta})=\mathbf{P}(\boldsymbol{\eta}(t))-\mathbf{P}(\mathrm{\eta})-\nabla\mathbf{P}(\mathrm{\eta})^\top(\boldsymbol{\eta}(t)-\mathrm{\eta}). \qquad (19)$$

Note that (19) is the *Bregman distance*, see Bregman [1967], associated with $\mathbf{P}$ between $\boldsymbol{\eta}(t)$ and $\mathrm{\eta}$. The function $B_P(\boldsymbol{\eta}(t),\mathrm{\eta})$ is positive definite and radially unbounded since $P(\cdot)$ is a strictly convex function. From (18) it follows now

$$\dot{\mathbf{B}}_P(\boldsymbol{\eta}(t),\mathrm{\eta})=-(\boldsymbol{y}(t)-\mathbf{y})^\top(\boldsymbol{u}(t)-\mathbf{u}). \qquad (20)$$

We can use these observations now for a Lyapunov analysis of the closed-loop system.

Theorem 5.1. Consider the network of dynamical systems (1), with the control inputs defined in (15). Let Assumption 2.2 and Assumption 4.1 hold. Then the network (1), (13) converges to the agreement steady-state $\mathbf{y}=\beta\mathbf{1}$, and

$$V(\boldsymbol{x}(t),\boldsymbol{\eta}(t))=\mathbf{S}(\boldsymbol{x}(t))+\mathbf{B}_P(\boldsymbol{\eta}(t),\mathrm{\eta}) \qquad (21)$$

with $\mathbf{S}(\boldsymbol{x}(t)):=\sum_{i=1}^{|\mathbf{V}|}S_i(x_i(t))$ (i.e. the EIP storage functions), is a Lyapunov function for the closed-loop system.

Proof. The Lyapunov function $V(\boldsymbol{x}(t),\boldsymbol{\eta}(t))$ is positive definite since both $S_i(x_i(t))$ and $\mathbf{B}_P(\boldsymbol{\eta}(t),\mathrm{\eta})$ are positive definite. By assumption, all system (1) are OSEIP and thus

$$\dot{\mathbf{S}}(\boldsymbol{x}(t))\leq-\gamma\|\boldsymbol{y}(t)-\mathbf{y}\|^2+(\boldsymbol{y}(t)-\mathbf{y})^\top(\boldsymbol{u}(t)-\mathbf{u}). \qquad (22)$$

Thus, combining (22) and (20), we obtain the directional derivative of the Lyapunov function candidate as

$$\dot{V}(\boldsymbol{x}(t),\boldsymbol{\eta}(t))=\dot{\mathbf{S}}(\boldsymbol{x}(t))+\dot{\mathbf{B}}_P(\boldsymbol{\eta}(t),\mathrm{\eta})\leq-\gamma\|\boldsymbol{y}(t)-\mathbf{y}\|^2.$$

Since $\lim_{t\to\infty}\ V(\mathbf{x}(t))-V(\mathbf{x}(t_0))$ exists and is finite, and since ρ_i are positive definite, we conclude from Barbalat's lemma (Khalil [2002]) that $\lim_{t\to\infty}\|y_i(t)-\mathrm{y}_i\|=\lim_{t\to\infty}\|y_i(t)-\beta\|\to 0$. Additionally, by the invertability of the input-to-output map follows that $\boldsymbol{u}(t)$ converges to $\mathbf{u}$ and, consequently, $\boldsymbol{\eta}(t)$ converges to $\mathrm{\eta}$.

We can summarize the observations of this section as follows. All signals in the dynamical network (1), (9), (15), (8) converge to their steady-state, and their steady-state values are optimal solutions to some network optimization problems, defined by the node dynamics or the chosen controller, respectively. We want to emphasize that this network interpretation revealed a duality relation between the plant outputs and inputs, as well as between the controller states and the controller inputs.

6. OPTIMAL DISTRIBUTION CONTROL

We discuss now the previous results on a specific example. We consider therefore the flow control problem in a multi-inventory system. A similar discussion has been presented Bürger and De Persis [2013], where optimal inventory control is studied from an internal model control perspective. The contribution of this paper is the presentation of the results in the context of the network optimization problems discussed above. The routing control in multi-inventory systems has also been studied in Bauso et al. [2006], De Persis [2013].

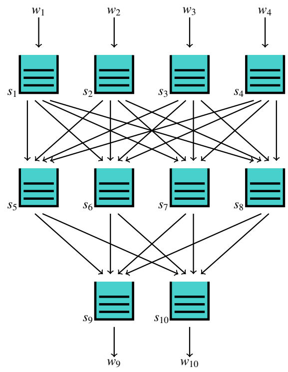

Fig. 2. Illustration of a 10-inventory system. Iinventories $i=1$–4 are supplied with the amount of goods $w_i>0$, while there is a constant demand at nodes $i=9,10$, with $w_i<0$.

We consider here the problem of *optimal routing* in a distribution network with several inventories, with a deteriorating storage, see e.g. Goyal and Giri [2001] for a justification of this model. Consider n inventories with inventory levels $I_i(t)$. The inventory level is influenced by the external supply or demand to this inventory $D_i(t)$, the amount of goods shipped to/from another inventory R_i and a decay rate $\theta_i>0$, modeling the perishing of goods in the inventory. This leads to the dynamics of one inventory system as

$$\frac{dI_i}{dt}=D_i(t)+R_i(t)-\theta_iI_i(t),\quad i=1,\dots,n. \qquad (23)$$

We suppose here that the external demand or supplies at one node are constant, i.e., $D_i(t)=D_i$ and that it is balanced over the network, i.e., $\sum_{i=1}^n D_i=0$. Goods can be shipped between inventories along m transportation lines of the network. Let E be the incidence matrix, describing the incidence relation of transportation lines and inventories, then we have

$$\mathbf{R}(t)=E\boldsymbol{\mu}(t), \qquad (24)$$

where $\mathbf{R}(t)=[R_1(t),\dots,R_n(t)]^\top$ and $\boldsymbol{\mu}(t)$ is the amount of goods transported in the network. A schematic illustration of such an inventory system with 10 inventories is shown in Figure 2.

We additionally assume that transporting goods within the network incorporates a costs, characterized by the convex function

$$\mathcal{F}_k(\mu_k),\quad k=1,\dots,m. \qquad (25)$$

It is reasonable to assume that the transportation capacity of one line is limited. Therefore, we might constrain the flows on one line to the set $\Gamma_k=\{\mu_k:-w_k\leq\mu_k\leq w\}$ for some capacity bound $w_k>0$. This constraint can be integrated into the flow cost functions by simply defining $\mathcal{F}_k$ such that

$$\mathcal{F}_k(\mu_k)\begin{cases}<\infty & \text{if } \mu_k\in\Gamma_k\\ =+\infty & \text{if } \mu_k\notin\Gamma_k.\end{cases}$$

However, we assume in the following that $\mathcal{F}_k$ is twice continuously differentiable and $\nabla^2\mathcal{F}_k(\mu_k)>0$ for all $\mu_k\in\Gamma_k$.

We assume in the following that the optimal routing problem is feasible, and that there exists $\boldsymbol{\mu}\in\mathrm{int}(\Gamma_1)\times\cdots\times\mathrm{int}(\Gamma_m)$, with $\mathrm{int}(\Gamma_k)$ being the interior of Γ_k, such that $\mathbf{D}+E\boldsymbol{\mu}=0$, where $\mathbf{D}=[D_1,\dots,D_n]^\top$.

We aim now to design a *distributed control law* that takes only the imbalance between neighboring inventory systems as inputs and regulates the inventory system to a steady-state configuration where the following two objectives are satisfies: (i) all storage levels are balanced (see e.g. De Persis [2013]), and (ii) the flow minimizes the cost induced by the cost functions (25).

The network theoretic interpretation of the passivity-based cooperative control system presented above will provide us directly with a solution to this control problem. First, we note that the dynamics of each inventory system (23) is equilibrium independent passive if we take the input $u_i(t) = R_i(t)$ and the output $y_i(t) = I_i(t)$. The equilibrium input to output map is then $k_{y,i}(\mathrm{u}_i) = \frac{1}{\theta_i}(\mathrm{u}_i + D_i)$. The optimization problem (OFP1), which is used to determine the equilibrium inputs is now easily formulated as

$$\min_{\mathbf{u}\in\mathcal{R}(E)} \sum_{i=1}^{n} \left(\frac{1}{2\theta_i}\mathrm{u}_i^2 + \frac{1}{\theta} D_i \mathrm{u}_i \right).$$

The dual problem (OPP1) can also be easily derived as

$$\min_{\mathbf{y}\in\mathcal{N}(E^\top)} \sum_{i=1}^{n} \left(\frac{\theta}{2}\mathrm{y}_i^2 - D_i \mathrm{y}_i \right)$$

From the latter follows directly that the agreement steady state output is $\mathbf{y} = 0$, i.e., the all zeros vector. The agreement steady state input is then $\mathbf{u} = \mathbf{D}$.

The (static) optimal distribution problem, we aim to solve with the feedback controller, is the following

$$\min_{\mu} \sum_{k=1}^{m} \mathcal{F}_k(\boldsymbol{\mu}_k) \;\; \text{s.t. } D + E\mu = 0, \tag{26}$$

i.e., the controller should route the optimal supply or demand instantaneously through the network. We can compare now the problem (OFP2), and directly see that the two problems are identical with $P_k^\star(\boldsymbol{\mu}_k) = \mathcal{F}_k(\boldsymbol{\mu})$.

A direct consequence of our previous discussion is, thus, that the distributed feedback controller

$$\begin{aligned} \dot{\boldsymbol{\eta}}(t) &= E^\top \boldsymbol{I} \\ \boldsymbol{R} &= -E\nabla\boldsymbol{\mathcal{F}}^{-1}(\boldsymbol{\eta}(t)) \end{aligned} \tag{27}$$

with $\boldsymbol{I} = [I_1, \ldots, I_n]^T$, and $\boldsymbol{\mathcal{F}}^{-1}(\boldsymbol{\eta}(t)) = \sum_{k=1}^{m} \mathcal{F}_k^{-1}(z_k(t))$, solves the optimal routing problem. Note that the proof of Theorem (5.1) remains valid, since, under the stated feasibility assumption the Bregman distance $\mathbf{B}_{\mathcal{F}^\star}$ is a positive definite function and can serve as a Lyapunov function.

As an example, consider the integral function of $w_k \tanh^{-1}(\mu_k)$ as a C^2 function that is finite only within the capacity constraints; that is $\mathcal{F}_k(\mu_k) = \frac{1}{2} w_k \log(1 - \mu_k^2) + w_k \mu_k \tanh^{-1}(\mu_k)$ and $\nabla\mathcal{F}^{-1}(z(t)) = w_k \tanh(z(t))$. In this case, the routing along a transportation line will always, even during the transient behavior, satisfy the capacity bounds.

7. CONCLUSIONS

We studied in this paper the inverse optimality properties of the steady-states in passivity based cooperative control problems. We considered a canonical feedback control structure involving equilibrium independent passive systems on the network nodes and dynamical controllers on the networks edges. We have shown that the output-agreement steady state is fully defined by a dual pair of network optimization problems. This result revealed also a duality relation between the inputs and the outputs. We have then shown that an internal model controller, which is able steer the system to output agreement, has also a direct connection to a dual pair of network optimization problems. These optimization problems explained a duality relation between the controllers state and the controllers output. We could then exploit the duality relation between the different variables to derive a Lyapunov function for the closed-loop system.

The presented results open a novel perspective on passivity-based cooperative control and explain certain inverse optimality properties. However, beyond that, the results have implications also for controller design, as we illustrate on the optimal routing control problem in an inventory system.

REFERENCES

M. Arcak. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 52(8):1380–1390, 2007.

H. Bai, M. Arcak, and J. Wen. *Cooperative control design: A systematic, passivity–based approach*. Springer, New York, NY, 2011.

D. Bauso, F. Blanchini, and R. Pesenti. Robust control strategies for multiinventory systems with average flow constraints. *Automatica*, 42(8):1255 – 1266, 2006. ISSN 0005-1098.

L.M. Bregman. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. *USSR Computational Mathematics and Mathematical Physics*, 7(3):200 – 217, 1967. ISSN 0041-5553. doi: 10.1016/0041-5553(67)90040-7.

M. Bürger and C. De Persis. Internal models for nonlinear output agreement and optimal flow control. In *IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, 2013. to appear.

M. Bürger, G. S. Schmidt, and F. Allgöwer. Preference based group agreement in cooperative control. In *Proc. of 8th IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, pages 149 – 154, Bologna, Italy, 2010.

M. Bürger, D. Zelazo, and F. Allgöwer. Network clustering: A dynamical systems and saddle-point perspective. In *Proc. of IEEE Conference on Decision and Control*, pages 7825–7830, Orlando, Florida, Dec. 2011.

M. Bürger, D. Zelazo, and F. Allgöwer. Hierarchical clustering of dynamical networks using a saddle-point analysis. *IEEE Transactions on Automatic Control*, 58(1):113 – 124, 2013. doi: 10.1109/TAC.2012.2206695.

N. Chopra and M. Spong. *Advances in Robot Control, From Everyday Physics to Human-Like Movements*, chapter Passivity-based Control of Multi-Agent Systems, pages 107–134. Springer, 2006.

C. De Persis. Balancing time-varying demand-supply in distribution networks: an internal model approach. In *Proc. of European Control Conference*, 2013. to appear.

C. De Persis and B. Jayawardhana. Coordination of passive systems under quantized measurements. *SIAM Journal on Control and Optimization*, 50 (6):3155 – 3177, 2012.

P. DeLellis, M. di Bernardo, and G. Russo. On QUAD, lipschitz, and contracting vector fields for consensus and synchronization of networks. *IEEE Transactions On Circuits and Systems - I*, 58(3):576 – 573, 2011.

C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, 2001.

S. K. Goyal and B.C. Giri. Recent trends in modeling of deteriorating inventory. *Eurpean Journal of Operational Research*, 134:1–16, 2001.

G. H. Hines, M. Arcak, and A. K. Packard. Equilibrium-independent passivity: A new definition and numerical certification. *Automatica*, 47:1949 – 1956, 2011.

H. Khalil. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, New Jersey, 2002.

R. T. Rockafellar. *Network Flows and Monotropic Optimization*. Athena Scientific, Belmont, Massachusetts, 1998.

L. Scardovi, M. Arcak, and E. D. Sontag. Synchronization of interconnected systems with applications to biochemical networks: An input-output approach. *IEEE Transactions on Automatic Control*, 55(6):1367–1379, 2010.

G.-B. Stan and R. Sepulchre. Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*, 52(2):256 – 270, 2007.

A. J. van der Schaft and B. M. Maschke. Port-hamiltonian systems on graphs. Sep. 2012. arXiv:1107.2006v2 [math.OC].

P. Wieland, R. Sepulchre, and F. Allgöwer. An internal model principle is necessary and sufficient for linear output synchronization. *Automatica*, 47: 1068 – 1074, 2011.

D. Zelazo and M. Mesbahi. Edge agreement: Graph-theoretic performance bounds and passivity analysis. *IEEE Transactions on Automatic Control*, 56 (3):544–555, 2010. ISSN 0018-9286.