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Combinatorial Insights and Robustness Analysis for Clustering in Dynamical Networks

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***Abstract*— This paper studies a clustering phenomena that emerges from a dynamic network with bounded and non-linear interactions. Necessary and sufficient conditions are given describing when the network exhibits clustering. We introduce a *synchronization coefficient* to quantify whether a network is synchronizing or clustering and provide a robustness margin for clustering. A combinatorial description of the dynamic network clustering is provided that relates to optimal graph partitioning. Finally, the synchronization coefficient is also used for defining a set of critical disturbances that can cause the system to cluster.**

I. Introduction

Synchronization is an important emerging behavior seen in many complex dynamical networks. Originally observed as a natural phenomenon in physics, biology, and neuroscience, synchronization has also found applications in engineering systems such as with group coordination or the synchronization of power networks [1], [2]. This, in turn, has led to a new notion of stability and currently an advanced control theory for synchronization is being developed [3], [4].

Another emergant behavior of complex networks is clustering or cluster synchronization. Clustering is a phenomenon where subgroups of agents within a complex system agree on a common state. This kind of behavior has been observed across various disciplines ranging from social networks and opinion dynamics to large-scale power networks [5], [6]. In general, the clustering phenomena of dynamical networks is considered a hard problem both for analysis and prediction. Moreover, in many engineered systems, clustering is deemed an undesirable behavior as opposed to a synchronous state (e.g., in power systems or flocking in multi-agent systems).

As a result, the development of a fundamental theory for clustering in dynamical networks is gaining attention. Various mathematical models with different mechanisms leading to clustering have been proposed in the literature. This includes the celebrate “bounded confidence opinion dynamics” [6], where clustering is caused by a state-dependent interaction graph [7]. Clustering in different diffusively coupled networks has been studied in [8], [9]. In [10], [11] a dynamical network model with bounded interaction rules is proposed. The authors of this paper presented in a previous work an alternative clustering model that also used a bounded interaction rule leading to a clustering behavior [12], [13]. The main contribution of this model was that the dynamic clustering behavior could be precisely analyzed and predicted by considering an associated *static* optimization problem.

This paper complements our previous work [12], [13] and contributes towards a theory for clustering in dynamical networks. In particular, we analyze the transition from a synchronous to a clustered behavior in dynamical networks. The first contribution of this paper is the introduction of the *synchronization coefficient* that quantifies a networks ability to synchronize. We show that the dynamical model can synchronize if and only if the synchronization coefficient does not exceed unity; otherwise, the system will exhibit clustering. The synchronization coefficient is shown to be strongly related to a purely combinatorial property of the underlying network graph. As the second contribution of this paper, it is shown that the network synchronizes if an only if all possible bi-partitions of the graph satisfy a specified partition quality criterion. Therefore, this paper establishes a connection between the combinatorial problem of *optimal graph partitioning* and *clustering in dynamical networks*. The synchronization coefficient is also used to characterize robustness properties of the synchronous state in the presence of disturbances. We provide sufficient conditions for disturbances entering both nodes and edges of the network that lead to clustering. This condition provides a characterization of the nodes and edges that can most impact the synchronous state in the presence of disturbances. We show that these critical nodes and edges are directly related to the synchronization coefficient. In this direction, this paper contributes towards a robustness theory for clustering networks.

The remainder of the paper is organized as follows. The dynamical network model is presented in §II. The synchronization coefficient is introduced in §III along with necessary and sufficient conditions for clustering. These conditions are then connected to optimal partitions of the interaction graph in §IV. In §V we characterize critical disturbances leading to clustering in a given network. Finally, some concluding remarks are given in §VI.

Preliminaries: Throughout this paper we consider systems defined over graphs [14]. A *graph*, $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, consists of a set of *nodes*, $\mathbf{V} = \{v_1, \dots, v_n\}$, and a set of *edges*, $\mathbf{E} = \{e_1, \dots, e_m\}$ describing the incidence relation between pairs of nodes. We assume *connected* graphs for this work. We make use of the *incidence matrix*, $E(\mathcal{G}) \in \mathbb{R}^{|\mathbf{V}|\times|\mathbf{E}|}$ of $\mathcal{G}$, defined in the standard way after assigning an arbitrary orientation to each edge [14].

For a given graph $\mathcal{G}$ a *cut-set* is a set of edges whose deletion leads to an increase in the number of connected components in $\mathcal{G}$. A cut-set always induces a partition of the nodes. A *p-partition* of $\mathcal{G}$ is a collection of node sets $\mathbb{P} = \{\mathbf{P}_1, \dots, \mathbf{P}_p\}$ with $\mathbf{P}_i \subseteq \mathbf{V}$, $\cup_{i=1}^{p} \mathbf{P}_i = \mathbf{V}$, and $\mathbf{P}_i \cap \mathbf{P}_j = \emptyset$ for all $\mathbf{P}_i, \mathbf{P}_j \in \mathbb{P}$, such that each *subgraph* $\mathcal{P}_i$ induced by the node sets $\mathbf{P}_i$ is connected. Throughout this paper, we follow the convention that bold-faced capital letters refer to sets, as in $\mathbf{V}$, and the script notation for graphs, as in $\mathcal{P}$.

II. A Dynamic Model for Clustering

We study a class of dynamical networks that exhibit asymptotically a clustered behavior. The mathematical model of this network was originally presented and analyzed in [12], [13]. A dynamic state variable $x_i(t)$ is assigned to each node $v_i \in \mathbf{V}$, and $z_k(t)$ is assigned to each edge $e_k \in \mathbf{E}$. The proposed dynamic network has the form

$$\begin{aligned} \dot{x} &= -\nabla \mathbf{J}(x) - E(\mathcal{G}) W \boldsymbol{\psi}(z) \\ \dot{z} &= E(\mathcal{G})' x, \end{aligned} \tag{1}$$

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where $x = [x_1, \ldots, x_n]'$ and $z = [z_1, \ldots, z_m]'$. The (possibly nonlinear) vector valued function $\nabla \mathbf{J}(x) = [\nabla J_1(x_1), \ldots, \nabla J_n(x_n)]'$ is defined on the nodes and $\nabla J_i(x_i)$ are the gradients of *strictly convex* functions $J_i(x_i)$. The functions $\boldsymbol{\psi}(z) = [\psi_1(z_1), \ldots, \psi_m(z_m)]'$ are assigned to the edges and represent the nonlinear interaction between the nodes. We assume that the functions $\psi_k(z_k)$ are monotonically increasing and *ultimately bounded*, i.e.,

$$\lim_{z_k \to \infty} \psi_k(z_k) = 1, \quad \lim_{z_k \to -\infty} \psi_k(z_k) = -1.$$

The edge weighting matrix $W \in \mathbb{R}^{m \times m}$ is a diagonal matrix $W = \mathrm{diag}(\alpha_1, \ldots, \alpha_m)$, where the α_k's are referred to as the *edge capacities* of the edges.

Definition 2.1: A network $\mathcal{G}$ is *synchronizing* if $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$ for all $v_i, v_j \in \mathbf{V}$.
When a network is synchronizing, each state converges to what is termed the *agreement state*, $x^* \in \mathbb{R}$; that is $\lim_{t\to\infty} x_i(t) \to x^*$ for all i. A network might not be synchronizing but still exhibit a structured asymptotic behavior, in which groups of nodes synchronize.

Definition 2.2: A network is *clustering* if there exists a p-partition $\mathbb{P}$ such that $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$ for all $v_i, v_j \in \mathbf{P}_l, l \in \{1, \ldots, p\}$.

Clustering Structure and Saddle-Point Analysis

A main result from [13] is that the system (1) is clustering in its asymptotic behavior. The exact structure of the clustering was shown to depend on (i) the functions $\nabla J_i(x_i)$, (ii) the interaction graph $\mathcal{G}$ and (iii) the edge capacities α_k, but is independent of the initial conditions and the form of the nonlinear functions $\psi_k(z_k)$. The main analytic tool for characterizing the clustering behavior of (1) was to relate the asymptotic solution to the solution of a corresponding *static saddle-point problem*,

$$\max_{\mu\in\Gamma} \min_{\mathrm{x}} \quad \sum_{i=1}^{n} J_i(\mathrm{x}_i) + \mu E(\mathcal{G})'\mathrm{x}, \tag{2}$$

where $\Gamma = \Gamma_1 \times \ldots \times \Gamma_m$ with $\Gamma_k = \{\mu_k : -\alpha_k \leq \mu_k \leq \alpha_k\}$. Note that the decision variables $\mathrm{x} \in \mathbb{R}^n$ are associated with the nodes, while the variables $\mu \in \mathbb{R}^m$ are associated with the edges. We denote the set of saddle-point solutions to (2) by $(\mathbb{X}, \mathbb{M})$. It was shown in [12] that the x solution to the saddle-point problem is unique, while the set $\mathbb{M}$ might contain infinitely many points. In particular, the set $\mathbb{M}$ is related to the *flow space* of $\mathcal{G}$ [14], and can be expressed as $\mathbb{M} = \{\mu \in \Gamma | \mu = \mu^* + \nu, \nu \in \mathcal{N}(E(\mathcal{G})\}$ for some $\mu^* \in \mathbb{M}$.

The following result relates the saddle-point problem (2) to the dynamical network (1).

Theorem 2.3 ([12]): Let $\mathbb{X} \times \mathbb{M}$ be the set of saddle-points of problem (2). The trajectories $x(t)$ of (1) remain bounded and $\lim_{t\to\infty} x(t) \to \mathbb{X}, \quad \lim_{t\to\infty} W\boldsymbol{\psi}(z(t)) \to \mathbb{M}$.

The implications of this result is that analysis of the network clustering can be considered from a static convex optimization standpoint instead of a dynamic and nonlinear systems perspective. We now recall some of the main ideas developed in [12] required for this work. An important concept for the analysis of clustering structures is the notion of *saturated edges*.

Definition 2.4: An edge $e_k \in \mathbf{E}$ is said to be *saturated* if for all $\mu^* \in \mathbb{M}$, $\mu_k^* \in \partial\Gamma_k$ (e.g., $|\mu_k^*| = \alpha_k$).
Saturated edges are edges for which the constraint is active but cannot be varied within the set $\mathbb{M}$. Since $\mathbb{M}$ is related to the cycles of $\mathcal{G}$, the saturated edges must always partition the graph.

Lemma 2.5 ([12]): The set of saturated edges in $\mathbb{M}$ forms a cut-set for the graph.
This implies that any (undirected) cycle in $\mathcal{G}$ either contains none or at least two saturated edges, but cannot contain a single saturated edge. Saturated edges are crucial for the clustering structure of the network.

Theorem 2.6 ([12]): Let $\mathbb{X} \times \mathbb{M}$ be the saddle-points of (2), and let $\mathbf{Q} \subseteq \mathbf{E}$ be the set of saturated edges. Then $\mathbf{Q}$ induces a p-partition $\mathbb{P} = \{\mathbf{P}_1, \ldots, \mathbf{P}_p\}$ and the network is clustering according to $\mathbb{P}$.

With the results of Theorem 2.3 and Theorem 2.6 the clustering structure of the dynamical model (1) can be exactly characterized using the saddle-point problem (2).

III. The Synchronization Coefficient and the Clustering Margin

The previous analysis of Theorem 2.6 shows that the network (1) is either synchronizing or clustering. In this section we derive necessary and sufficient conditions for (1) to exhibit clustering. We will focus on the transition from a synchronous to a clustering behavior and exploit the properties of the saddle-point solutions to derive these conditions.

We first consider the situation that the network is synchronizing. Due to Theorem 2.3, the agreement state x^* of the network (1) is independent of the initial condition and is only determined by the properties of the network. We can conclude from Theorem 2.6 that the network reaches agreement if no edges are saturated. This implies that the network will always reach agreement if the edge capacities α_k are sufficiently large. Consequently, we are able to relate the agreement state to the solution of a static *network optimization problem*,

$$\min_{\mathrm{x}_i} \sum_{i=1}^{n} J_i(\mathrm{x}_i) \text{ s.t. } E(\mathcal{G})'\mathrm{x} = 0. \tag{3}$$

The equality constraint forces all decision variables on the nodes x_i to be identical. Note that for $\Gamma = \mathbb{R}^m$ the saddle-point problem (2) is the Lagrage dual of (3).

Proposition 3.1: The agreement state x^* for the system (1) that is synchronizing is the unique minimizer of (3).

Proof: The trajectories $x(t)$ converge to the unique solution of the saddle point problem (2). For sufficiently large edge capacities, the primal and dual solution to the network optimization problem (3) is also a solution to the saddle-point problem. Since the x solution of the saddle point problem is unique, the network must agree on x^*. ∎

A main contribution of this section is the proposal of a measure which indicates whether a network is synchronizing or clustering. We therefore introduce the notion of a *synchronization coefficient* for a given network (1) with agreement state x^*.

Definition 3.2: The synchronization coefficient γ^* of a network with form (1) is

$$\gamma^* := \min_{\mu} \|W^{-1}\mu\|_\infty \quad \text{s.t. } \nabla\mathbf{J}(\mathrm{x}^*) + E(\mathcal{G})\mu = 0. \tag{4}$$

The value γ^* contains important information about the synchronization or clustering structure of the network.

Theorem 3.3: The dynamical network (1) is synchronizing if and only if $\gamma^* \leq 1$.

Proof: We first show that $\gamma^* \leq 1$ implies that all nodes converge to x^*. If $\gamma^* \leq 1$, there is a μ^* such that $|\mu_k^*| \leq \alpha_k$ for all k and thus $\mu^* \in \Gamma$. This vector μ^* is a dual solution to (3) and satisfies the first-order optimality condition

$$\nabla\mathbf{J}(\mathrm{x}^*) + E(\mathcal{G})\mu^* = 0. \tag{5}$$

With the two conditions $\mu^* \in \Gamma$ and and μ^* satisfying (5), we can conclude that (x^*, μ^*) is a saddle-point solution to (2). It now follows from Theorem 2.3 that all trajectories $x_i(t)$ of (1) will asymptotically converge to x^* and therefore synchronize.

It remains to show that if the network synchronizes then $\gamma^* \leq 1$. If the network synchronizes, from the uniqueness of x^*, there must be a saddle point solution (x^*, μ^*) satisfying (5) with x^* being the agreement state and $\mu^* \in \Gamma$. This implies that $|\mu_k^*| \leq \alpha_k$ and thus, $\gamma^* \leq 1$. ■

Theorem 3.3 allows to quickly verify whether the network will synchronize. The condition is a linear program and can be checked efficiently. The synchronization coefficient has an interpretation, relating to clustering.

Corollary 3.4: The dynamical network (1) is clustering if and only if $\gamma^* > 1$.

The synchronization coefficient γ^* can be interpreted as a *robustness measure* of the network.

Definition 3.5: The *clustering robustness margin* for a network in synchronization is $1 - \gamma^*$.

The synchronization coefficient contains information about how much the network can be modified before it is clustering. Another important observation is that the value γ^* is always determined by a cut-set of $\mathcal{G}$.

Proposition 3.6: Let μ^* be a solution to (4) and let $\mathbf{Q}$ be the set of edges for which $|\mu_k^*| = \gamma^* \alpha_k$. Then $\mathbf{Q}$ is a cut-set.

Proof: Given any solution μ^* satisfying the equality constraint of (4), any other saddle point $\tilde{\mu} \in \mathbb{M}$ must also satisfy the constraint. The null-space of the incidence matrix $\mathcal{N}(E(\mathcal{G}))$ contains all signed path vectors $\zeta \in \{-1, 0, 1\}^m$ corresponding to the cycles of $\mathcal{G}$ (see e.g., [12] or [14]). Therefore, one can always find a $\tilde{\mu}$ such that for a particular edge e_k, contained in a cycle $\mathcal{C}$, $|\tilde{\mu}_k| < |\mu_k^*|$. However, there must then be at least one other edge e_l in the cycle $\mathcal{C}$ for which $|\tilde{\mu}_l| > |\mu_l^*|$, a consequence of Lemma 2.5.

The result is now demonstrated via contradiction. Assume that the set of edges $\mathbf{Q}$, with $|\mu_k^*| = \gamma^* \alpha_k$ for $e_k \in \mathbf{Q}$, does not form a cut-set. Then every edge $e_k \in \mathbf{Q}$ must be contained in at least one undirected cycle of $\mathcal{G}$, say $\mathcal{C}$, such that $(\mathcal{C} \setminus \{e_k\}) \cap \mathcal{Q} = \emptyset$. This, however, implies that one can find a δ, with $|\delta|$ sufficiently small, and define $\tilde{\mu} = \mu^* + \delta z$, where z is the signed path vector corresponding to the cycle $\mathcal{C}$, such that $|\tilde{\mu}_k| < |\mu_k^*| = \gamma^* \alpha_k$ for $e_k \in \mathbf{Q}$ and $|\tilde{\mu}_l| < \gamma^* \alpha_l$ for all other edges e_l in the cycle $\mathcal{C}$. This contradicts the original assumption that $\gamma^* = \min \|W^{-1}\mu\|$, and thus $\mathbf{Q}$ must form a cut-set. ■

We will refer to $\mathbf{Q}$ in the following as the *γ^*-cut set*. The synchronization coefficient γ^* and the γ^*-cut set have various implications on the synchronization and clustering behavior of (1). These quantities also have a combinatorial interpretation that we explore next.

IV. A Combinatorial Condition for Clustering

The results of §III related the synchronization coefficient γ^* to the dynamic behavior of (1). We show that this condition is also intimately connected to a purely combinatorial property. The combinatorial problem of finding an optimal graph partition is related to the clustering behavior of (1).

We assume that the agreement state x^* and the edge capacities α_k are known. We can then interpret this network as a *static weighted graph* with $\nabla J_i(\mathrm{x}^*)$ being node-weights and α_k being edge-weights. Note that due to the optimality properties of x^*, it holds that $\sum_{i=1}^n \nabla J_i(\mathrm{x}^*) = 0$.

We consider in the following only 2-partitions of the (weighted) graph $\mathcal{G}$. The set of all possible 2-partitions of $\mathcal{G}$ is denoted by $\mathbb{P}_2(\mathcal{G})$. A particular 2-partition $\mathbb{P} \in \mathbb{P}_2(\mathcal{G})$ is then characterized by the triplet $\mathbb{P} = (\mathbf{P}_1, \mathbf{P}_2, \mathbf{Q})$, where $\mathbf{P}_1$ and $\mathbf{P}_2$ are disjoint node sets, such that $\mathbf{P}_1 \cup \mathbf{P}_2 = \mathbf{V}$ and $\mathbf{Q} \subset \mathbf{E}$ is the set of edges connecting the two node sets. We define the *quality* of a 2-partition in the following.

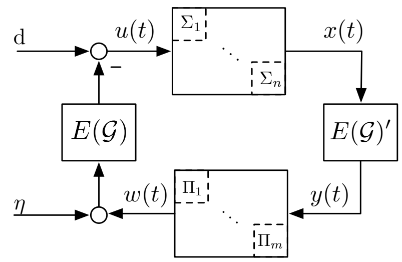

Fig. 1. Network model (1) with disturbances on the nodes and edges. The operator Σ_i maps the control $u_i(t)$ to the state $x_i(t)$; the operator Π_i maps the relative state $E(\mathcal{G}'x(t)$ to the edge state $w_k(t)$ [13].

Definition 4.1: The *quality* of a 2-partition $\mathbb{P} = (\mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}) \in \mathbb{P}_2(\mathcal{G})$ is

$$\Psi(\mathbb{P}) = \frac{|\sum_{i \in \mathbf{P}_1} \nabla J_i(\mathrm{x}^*)|}{\sum_{k \in \mathbf{Q}} \alpha_k}. \tag{6}$$

The quality of a graph partition is a purely combinatorial property. The quantity $|\sum_{i \in \mathbf{P}_1} \nabla J_i(\mathrm{x}^*)|$ can be interpreted as the weighted *imbalance* of the two clusters $\mathbf{P}_1$ and $\mathbf{P}_2$, since $\sum_{i \in \mathbf{P}_1} \nabla J_i(\mathrm{x}^*) = -\sum_{i \in \mathbf{P}_2} \nabla J_i(\mathrm{x}^*)$. Thus, the quality $\Psi(\mathbb{P})$ measures the ratio of the weight imbalance of the partitions and the capacity of the corresponding cut-set $\sum_{k \in \mathbf{Q}} \alpha_k$. Surprisingly, the measure $\Psi(\mathbb{P})$ is directly connected to the synchronization coefficient.

Theorem 4.2: Let $\mathbb{P}_2(\mathcal{G})$ be the set of all possible 2-partitions of $\mathcal{G}$ and let γ^* be the synchronization coefficient (4). Then

$$\max_{\mathbb{P} \in \mathbb{P}_2(\mathcal{G})} \Psi(\mathbb{P}) = \gamma^*. \tag{7}$$

The proof is presented in the appendix.

Thus, the quality of a partition provides a necessary and sufficient condition for synchronization of the system (1).

Theorem 4.3: The dynamical network (1) is synchronizing if and only if for all $\mathbb{P} \in \mathbb{P}_2(\mathcal{G})$, $\Psi(\mathbb{P}) \leq 1$.

It is also useful to interpret the previous result as a condition on the clustering behavior of the dynamical network.

Corollary 4.4: The dynamical network (1) is clustering if and only if there exists a 2-partition $\mathbb{P} \in \mathbb{P}_2(\mathcal{G})$ for which $\Psi(\mathbb{P}) > 1$.

This result relates the dynamic behavior of (1) to the properties of the weighted graph partitions. This result has an intuitive interpretation: The existence of partitions with high quality (i.e., $\Psi(\mathbb{P}) > 1$) corresponds to the existence of weakly connected components of the graph. If the weighted graph contains such weakly connected components then the corresponding dynamical network will cluster.

V. A Robustness Measure for Synchronization

Given a network in synchronization, a natural question is *how robust is the system against clustering*. In particular, is it possible to quantify how certain disturbances can affect and disrupt the synchronous state of the network. We consider an infiltration scenario on the network, where constant disturbance signal entering either the nodes or the edges of the system, see Figure 1. For this discussion, we restrict our attention to the case of linear node dynamics of the form

$$\nabla J_i(x_i) = q_i(x_i - \xi_i), \; q_i \in \mathbb{R}_{>0}, \xi_i \in \mathbb{R}. \tag{8}$$

We define $\xi = [\xi_1, \ldots, \xi_n]'$ and $Q = \text{diag}(q_1, \ldots, q_n)$. To aid the subsequent discussions, we present an alternative formulation of the synchronization coefficient problem (4).

A. An LP Reformulation of the Synchronization Coefficient

Problem (4) can also be written as the *linear program*

$$\begin{aligned} \gamma^* \quad &:= \quad \min_{t \geq 0, \mu} \; t \qquad (9) \\ \text{s.t.} \quad &E(\mathcal{G})\mu = -Q(\mathrm{x}^*\mathbb{1} - \xi), \; -t\mathbb{1} \leq W^{-1}\mu \leq t\mathbb{1}. \end{aligned}$$

One can now express the Lagrangian of (9) as

$$\begin{aligned} \mathcal{L}_{LP} \quad = \quad & t + p'\left(E(\mathcal{G})\mu + Q(\mathrm{x}^*\mathbb{1} - \xi)\right) + \\ & \lambda_u'(W^{-1}\mu - t\mathbb{1}) + \lambda_l'(-W^{-1}\mu - t\mathbb{1}), \qquad (10) \end{aligned}$$

where $p \in \mathbb{R}^n$ is the Lagrange-multiplier for the equality constraint and $\lambda_u \in \mathbb{R}^m_{\geq 0}$ ($\lambda_l \in \mathbb{R}^m_{\geq 0}$) is the multiplier corresponding to $W^{-1}\mu \leq t\mathbb{1}$ ($W^{-1}\mu \geq -t\mathbb{1}$). Note that p is a variable associated with the *nodes* in the graph, whereas λ_l, λ_u are associated with the *edges* in the graph. The edges for which the inequality constraints are active are exactly the edges in the γ^*-cut set.

The dual of (9) is given as

$$\begin{aligned} \max_{p, \lambda_u, \lambda_l} \qquad & Q(\mathrm{x}^*\mathbb{1} - \xi)'p \qquad (11) \\ \text{s.t.} \qquad & WE(\mathcal{G})'p = -\lambda_u + \lambda_l, \; 1 - \lambda_u'\mathbb{1} - \lambda_l'\mathbb{1} \geq 0. \end{aligned}$$

The form of (11) allows us to immediately infer certain properties of the optimal solution. The *complementary slackness* condition for (11) states that $\lambda_u'(W^{-1}\mu^* - t^*\mathbb{1}) = 0$ ($\lambda_l'(-W^{-1}\mu^* - t^*\mathbb{1}) = 0$) for all optimal solutions μ^*, t^* of the linear program (9). In particular, for any edge k such that $|W^{-1}\mu_k^*| < \gamma^*$, i.e., an edge not contained in the γ^*-cut set, it must be the case that the associated multipliers are identically zero (i.e., $[\lambda_u]_k = [\lambda_l]_k = 0$). Otherwise, for the γ^*-cut set edges, those with $|W^{-1}\mu_k^*| = \gamma^*$, then either $[\lambda_u]_k > 0$ and $[\lambda_l]_k = 0$ or $[\lambda_l]_k > 0$ and $[\lambda_u]_k = 0$.

The conditions on λ_u and λ_l have direct implications on the structure of the optimal solution p of (11). Consider all the edges $e_k = (v_i, v_j)$ such that $|W^{-1}\mu_k^*| < \gamma^*$. The complementary slackness conditions imply that $[WE(\mathcal{G})'p]_k = 0$; therefore, for these edges one must have that $p_i = p_j$. On the other hand, for the γ^*-cut set either $[WE(\mathcal{G})'p]_k = -[\lambda_u]_k$ or $[WE(\mathcal{G})'p]_k = [\lambda_l]_k$.

The key feature from this analysis is that the variable p defines a partition of the network. The elements p_i that are identical to each other define a partition in the graph, and the edges between them correspond to a cut-set; these are precisely the edges in the γ^*-cut set defined in §III.

Yet another interpretation arises from the formulation of the dual problem in (11). Using the standard price interpretation of the Lagrange multipliers, we find that the multiplier p effectively assigns a price to each partition. Therefore, comparing the "prices" of the induced partitions gives an indication of the importance that particular partition has on the synchronization coefficient. We explore this interpretation further in the sequel.

B. Disturbance and Infiltration Model

As we are considering a scenario where a disturbance or infiltrator is trying to disrupt the synchrony of the system, we assume that the system is already in a steady state. Therefore, we can consider the disturbance as a perturbation to the first-order optimality condition of the saddle-point problem (2).

As shown in Figure 1, disturbances may enter directly on the nodes or on the edges of the network. We model the effect of the *node disturbance* $d \in \mathcal{D} \subseteq \mathbb{R}^n$ and the *edge disturbances* $\eta \in \Omega \subseteq \mathbb{R}^m$ as a variation of the first-order optimality condition to (2) in the form

$$Q(\tilde{\mathrm{x}}\mathbb{1} - \xi) + E(\mathcal{G})(\tilde{\mu} + \eta) + d = 0. \qquad (12)$$

Here we recall that the synchronous solution (x^*, μ^*) will in general be different from the solution of the perturbed problem (12), denoted as $(\tilde{\mathrm{x}}, \tilde{\mu})$. We will also at times refer to the combined disturbance as the signal $\nu = (d, \eta) \in \mathcal{D} \times \Omega$.

Observe that adding a constant disturbance to either the nodes or edges in the network can lead in general to two behaviors. The first is that the entire network remains in a synchronous state, with the agreement value possibly being shifted. The second behavior leads to clustering; that is the disturbances cause the system to break from its synchronous state and form clusters.

It is in general difficult to characterize disturbances that will cause the network to cluster. One has, for any possible disturbance ν, first to compute the new agreement state and then to solve the LP for determining $\tilde{\gamma}$. Such a proceeding is neither efficient nor provides insight into the problem. It seems to be more desirable to quantify critical disturbances in terms of the undisturbed network configuration. This allows, without extensive simulations, to quantify whether a network configuration is robust or can be easily disrupted. We aim to derive such conditions in the following.

The perturbed equilibrium condition in (12) can be used to determine the new steady-state synchronous value of the network.

Proposition 5.1: Given a disturbance $d \in \mathcal{D}$ and $\eta \in \Omega$ on the steady-state node and edge dynamics, the resulting agreement value is given as

$$\tilde{\mathrm{x}} = \mathrm{x}^* - (\mathbb{1}'Q\mathbb{1})^{-1}\mathbb{1}'d. \qquad (13)$$

Proof: The proof follows directly from the first order optimality conditions. ■

Observe that the new agreement value is only affected by disturbances directly on the nodes. The edge disturbances do not influence the agreement state. It should also be emphasized that whether or not the system actually *obtains* the new agreement value depends on if the disturbances lead to clustering, or if the system remains synchronous.

Note that as a consequence of Proposition 5.1 the constraint (12) can be expressed entirely as a function of the undisturbed solution x^*.

$$Q(\mathrm{x}^*\mathbb{1} - \xi) + E(\mathcal{G})(\tilde{\mu} + \eta) + Ad = 0, \qquad (14)$$

where $A = \left(I - (\mathbb{1}'Q\mathbb{1})^{-1}Q\mathbb{1}\mathbb{1}'\right)$. Observe that if $Q = I$, then A is the graph Laplacian matrix for the complete graph [14]. For general Q, we can therefore conclude that any disturbance effectively influences each node in the network.

We can now examine how the synchronization coefficient changes under the disturbance $\nu \in \mathcal{D} \times \Omega$, which is the solution of the following optimization problem,

$$\tilde{\gamma} := \min_{\mu \in \mathbb{R}^{|\mathcal{E}|}} \|W^{-1}\mu\|_\infty \qquad (15)$$

$$\text{s.t. } Q(\mathrm{x}^*\mathbb{1} - \xi) + E(\mathcal{G})(\mu + \eta) + Ad = 0. \qquad (16)$$

To aid in the analysis, we first introduce the Lagrangian function associated with (15),

$$\begin{aligned} L(\mu, p, \nu) = & \|W^{-1}\mu\|_\infty + p'\left(Q(\mathrm{x}^*\mathbb{1} - \xi) + E(\mathcal{G})\mu\right) + \\ & p'\left(E(\mathcal{G})\eta + Ad\right). \qquad (17) \end{aligned}$$

Here we have explicitly included the disturbance in the description of the Lagrangian function to better illustrate its effect on (15).

Note that the Lagrange multipliers $p \in \mathbb{R}^n$ in (17) are precisely the same multipliers from the dual reformulation of the program in (10).

Following our notational convention, the optimal Lagrange multipliers for the undisturbed problem (i.e., when $\nu = 0$) is denoted $p^* \in \mathbb{R}^n$, while the optimal multipliers for the disturbed problem are denoted as $\tilde{p}$. In general we have that $\tilde{p} \neq p^*$.

Observe that for the undisturbed case, one has $\gamma^* = L(\mu^*, p^*, 0)$, which is the synchronization coefficient defined in (4). Meanwhile, for any other non-zero disturbance ν one has $\tilde{\gamma} = L(\tilde{\mu}, \tilde{p}, \nu)$. Due to the saddle-point properties associated with the optimal primal and dual solutions of (15), one can obtain the following inequality statements,

$$L(\mu^*, p, 0) \leq L(\mu^*, p^*, 0) \leq L(\mu, p^*, 0), \quad \forall \mu, p \tag{18}$$

$$L(\tilde{\mu}, p, \nu) \leq L(\tilde{\mu}, \tilde{p}, \nu) \leq L(\mu, \tilde{p}, \nu), \quad \forall \mu, p. \tag{19}$$

Note that in particular, the inequalities in (19) also hold for $\nu = 0$. We are now prepared to formally state how the disturbance can affect the network. The following theorem provides a bound on the synchronization coefficients in terms of the disturbance ν and the multipliers p^* and $\tilde{p}$.

Theorem 5.2: Given the optimal value of the undisturbed and disturbed synchronization coefficients γ^* and $\tilde{\gamma}$, their difference is bounded as

$$p^{*'} \left(E(\mathcal{G})\eta + Ad\right) \leq \tilde{\gamma} - \gamma^* \leq \tilde{p}' \left(E(\mathcal{G})\eta + Ad\right). \tag{20}$$

Proof: We show the left inequality first. Observe that $L(\tilde{\mu}, p^*, \nu) = L(\tilde{\mu}, p^*, 0) + p^{*'} \left(E(\mathcal{G})\eta + Ad\right)$. Consider that

$$\begin{aligned} \tilde{\gamma} = L(\tilde{\mu}, \tilde{p}, \nu) \quad &\geq \quad L(\tilde{\mu}, p^*, 0) + p^{*'} \left(E(\mathcal{G})\eta + Ad\right) \\ &\geq \quad L(\mu^*, p^*, 0) + p^{*'} \left(E(\mathcal{G})\eta + Ad\right) \\ &= \quad \gamma^* + p^{*'} \left(E(\mathcal{G})\eta + Ad\right) \end{aligned}$$

The right inequality is derived in the same manner as above, and omitted for brevity. ■

C. Synchronization Robustness and Critical Infiltration

The inequality provided in (20) leads to a direct interpretation of how disturbances can affect the robustness of the system against clusters. In particular, considering the left side inequality, one has

$$\tilde{\gamma} \geq \gamma^* + p^{*'} \left(E(\mathcal{G})\eta + Ad\right). \tag{21}$$

Note that this provides a lower bound on the synchronization coefficient in terms of the disturbances ν, the nominal synchronization coefficient γ^*, and the nominal multipliers p^* from the program (15). As the quantities γ^* and p^* represent the steady-state nominal properties of the network and it can now be quantified how ν can be used to disrupt that state. A critical disturbance leading to network clustering is one which makes $\tilde{\gamma}$ greater than unity.

Theorem 5.3: Given a dynamical network (1) with linear node dynamics (8) in synchronization, any disturbance $\nu = (d, \eta)$ in the set

$$\mathbb{V} = \{(d, \eta) \in \mathcal{D} \times \Omega \,|\, p^{*'}(E(\mathcal{G})\eta + Ad)) > 1 - \gamma^*\}$$

will cause the network to cluster.

This formulation provides a characterization of critical disturbances only in terms of information related to the synchronous network configuration. Since the vector p^* can be determined for the network in synchronization independent of disturbances, the set of critical disturbances can be determined off-line. However, note that there might actually be other disturbances not in $\mathbb{V}$ that also cause clustering. Our analysis provides, therefore, only a sufficient conditions for the critical disturbances.

We now pose a family of infiltration problems, where an infiltrator aims to disrupt the network. The set $\mathcal{D} \times \Omega$ then is used to describe the precise nature of the disturbance caused by the infiltrator. These problems all take the following general form:

$$\max_{d,\eta} \quad p^{*'} \left(E(\mathcal{G})\eta + Ad\right) \tag{22}$$

$$\text{s.t. } d \in \mathcal{D},\, \eta \in \Omega. \tag{23}$$

The set $\mathcal{D} \times \Omega$ then is used to describe the precise nature of the disturbance. We aim to answer where, in terms of the network structure, a disturbance is most critical and can cause the network to cluster. We characterize the critical nodes and edges in terms of a maximal increase of the lower bound on the synchronization coefficient (21).

Node Infiltration: We would like to determine which single node in the network, when confronted with a disturbance, can cause the largest change in the synchronization coefficient and therefore lead to network clustering with minimal effort. We assume here no disturbances on the edges, $\eta = 0$. The disturbance set is therefore the set of unit coordinate vectors $e_i \in \mathbb{R}^n$, with 1 in the ith component, and zeros elsewhere; i.e., $\mathcal{D} = \{e_1, \ldots, e_n\}$. In this setting, problem (22) becomes a mixed-integer linear program.

However, a closer examination of this problem reveals that an analytic solution is already available. First, note that

$$p^{*'}A = p^{*'} \left(I - (\mathbb{1}'Q\mathbb{1})^{-1}Q\mathbb{1}\mathbb{1}'\right) = p^{*'} - \frac{\sum_{i=1}^n p_i^* q_i}{\sum_{i=1}^n q_i}\mathbb{1}'.$$

Therefore, the optimal solution of (22) is equivalent to

$$\max_i \quad \left[p^* - \frac{\sum_{i=1}^n p_i^* q_i}{\sum_{i=1}^n q_i}\mathbb{1}\right]_i.$$

Here we recall our discussion from §V-A that the multiplier p^* induces a partition in the network. In particular, the multipliers corresponding to nodes within the same partition are identical. The vector p^* can be interpreted as a "price" and each element of that partition is assigned the same price. Therefore, by simply searching for the partition that is most "expensive," one can obtain the nodes at which a disturbance is most critical.

Proposition 5.4: The critical nodes are all nodes within the *partition induced by the γ^*-cut set* which has the largest multiplier value p^*.

Edge Disturbance: We now consider an analogous situation to the above scenario for edges. We assume no disturbances on the nodes, and the edge disturbances take the form $\Omega = \{e_1, \ldots, e_{|\mathcal{E}|}\}$ with $e_i \in \mathbb{R}^{|\mathcal{E}|}$. As in the node case, we find the solution can be obtained by examining the term $E(\mathcal{G})'p^*$. The price interpretation of the multiplier p^* is again relevant. Observe that the kth element of $E(\mathcal{G})'p^*$ is either 0 or $p_i^* - p_j^*$ for $e_k = (v_i, v_j)$.

Proposition 5.5: The critical edges are precisely the edges in the *γ^*-cut set.*

The edges linking the partitions induced by p^* with the largest difference can therefore have the largest affect on the synchronization coefficient. Note that the the critical nodes and edges are determined by maximizing a lower bound on the synchronization coefficient. The γ^*-cut set gives both the critical nodes and the critical edges on which a disturbance maximally increases the lower bound. There might, however, be other disturbances which increase the synchronization coefficient, but do not affect the lower bound.

This analysis contributes towards the development of a robustness theory for complex dynamical networks. It provides constructive methods for determining the critical nodes and edges at which a disturbance can cause the system to cluster.

VI. Concluding Remarks

The phenomena of clustering in coupled multi-agent system is observed in many natural and engineered systems yet is not well understood. This work provided a step towards a general theory for explaining clustering. By showing that the steady-state of the dynamic model is equivalent to the solution of a static saddle-point problem, we are able to inherit many tools available from static optimization. This then allowed us to provide a condition for when the system will synchronize or form clusters. This condition was termed the *synchronization coefficient*, and was also shown to be intimately related to robustness properties of the synchronous state. This was further explored in an infiltration scenario, where attacks on the nodes and edges of the system are made in attempt to cause the system to cluster. The synchronization coefficient also was used to show how the notion of clustering also solves a purely combinatorial problem of finding an optimal partition of the graph.

Appendix

A. Proof of Theorem 4.2

Proof: First, we show that $\Psi(\mathbb{P}) \leq \gamma^*$ for all possible partitions. Given γ^*, there exists a solution to (4) μ^* satisfying $|\mu_k^*| \leq \alpha_k \gamma^*$ for all k and satisfying the first-order optimality condition (5). One can now choose any 2-partition $\mathbb{P} = (\mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}) \in \mathbb{P}_2(\mathcal{G})$, and define, without loss of generality, $\mathbf{P}_1$ to be such that $\sum_{i\in\mathbf{P}_1} \nabla J_i(\mathrm{x}^*) > 0$. For a 2-partition, we define an *indicator vector* $\zeta \in \{-1, 1\}^n$ such that $\zeta_i = +1$ if node $i \in \mathbf{P}_1$ and $\zeta_i = -1$ if node $i \in \mathbf{P}_2$. Note that $\zeta'\nabla\mathbf{J}(\mathrm{x}^*) = 2\sum_{i\in\mathbf{P}_1} \nabla J_i(\mathrm{x}^*)$. Multiplying (5) from the left with the indicator vector leads to the condition

$$\zeta'\nabla\mathbf{J}(\mathrm{x}^*) + \zeta' E(\mathcal{G})\mu^* = 2\sum_{i\in\mathbf{P}_1} \nabla J_i(\mathrm{x}_i) + \zeta' E(\mathcal{G})\mu^* = 0. \tag{24}$$

Given the indicator vector of a partition, define a new vector $c = \frac{1}{2}E(\mathcal{G})'\zeta$, which has a very characteristic structure

$$c_k = \begin{cases} +1 & \text{if edge } k \text{ originates in } \mathbf{P}_1 \\ -1 & \text{if edge } k \text{ terminates in } \mathbf{P}_1 \\ 0 & \text{if edge } k \notin \mathbf{Q}. \end{cases}$$

Now, the condition (24) can be written as $\sum_{i\in\mathbf{P}_1} \nabla J_i(\mathrm{x}_i) = -c'\mu^*$. As $|\mu_k^*| \leq \alpha_k\gamma^*$ for all k, we obtain the upper bound

$$-c'\mu^* \leq \sum_{k\in Q} \alpha_k \gamma^*.$$

This bound leads to $(\sum_{i\in\mathbb{P}_1} \nabla J_i(\mathrm{x}_i)) \leq \sum_{k\in Q} \alpha_k\gamma^*$ and therefore to the conclusion that

$$\Psi(\mathbb{P}) = \frac{\sum_{i\in\mathbb{P}_1} \nabla J_i(\mathrm{x}_i)}{\sum_{k\in Q} \alpha_k} \leq \gamma^*. \tag{25}$$

This last inequality has to hold for any possible two partition proving the first direction.

In the second step, we show that $\max_{\mathbb{P}\in\mathbb{P}_2(\mathcal{G})} \Psi(\mathbb{P}) \geq \gamma^*$. With the goal to arrive at a contradiction, we assume that $\max_{\mathbb{P}\in\mathbb{P}_2(\mathcal{G})} \Psi(\mathbb{P}) < \gamma^*$. Again, without loss of generality, let any 2-partition $\mathbb{P} = (\mathbf{P}_1, \mathbf{P}_2, \mathbf{Q})$ be such such that $\sum_{i\in\mathbf{P}_1} \nabla J_i(\mathrm{x}^*) > 0$. It follows from the assumption that

$$\frac{\sum_{i\in\mathbf{P}_1} \nabla J_i(\mathrm{x}^*)}{\sum_{k\in\mathbf{Q}} \alpha_k} < \gamma^* \Leftrightarrow \sum_{i\in\mathbf{P}_1} \nabla J_i(\mathrm{x}^*) < \gamma^* \sum_{k\in\mathbf{Q}} \alpha_k \tag{26}$$

for any possible 2-partition. Additionally, we know that there is a (x^*, μ^*) satisfying (5). We define, as in the first part of the proof, an indicator vector ζ for any 2-partition. Multiplying (5) from left with ζ allows us to conclude that

$$\sum_{i\in\mathbb{P}_1} \nabla J_i(\mathrm{x}^*) = -c'\mu^*. \tag{27}$$

Combining the two conditions (26) and (27) leads to the new condition

$$-c'\mu^* < \gamma^* \sum_{k\in\mathbf{Q}} \alpha_k; \tag{28}$$

this must hold for any possible 2-partition. As a consequence of (28), there cannot be a 2-partition $\mathbb{P}$ with a corresponding cut-set $\mathbf{Q}$ for which $\mu_k^* = -c_k\alpha_k\gamma^*$ for all $k \in \mathbf{Q}$. However, if there is no such cut-set $\mathbf{Q}$, then every edge $e_l \in \mathbf{E}$ for which $|\mu_l^*| = \alpha_l\gamma^*$ must be contained in at least one cycle of edges $\mathbf{C}$ for which every other edge $e_s \in \mathbf{C}\setminus\{l\}$ has $|\mu_s^*| < \alpha_s\gamma^*$. But this implies now that one can find another $\tilde{\mu}$ satisfying (5), given by $\tilde{\mu} = \mu^* + \delta z$, where δ is some (maybe small) constant and $z \in \mathbb{R}^m$ is the signed path vector corresponding to the cycle $\mathbf{C}$ ([14]), such that $|\tilde{\mu}_l| < \alpha_l\gamma^*$ for every edge $e_l \in \mathbf{C}$. But this contradicts the definition of γ^*, as being the minimal value satisfying $\gamma^* \leq \frac{|\tilde{\mu}_k|}{\alpha_k}$ for all edges e_k. This is the contradiction we were looking for, and we can conclude that $\max_{\mathbb{P}\in\mathbb{P}_2(\mathcal{G})} \Psi(\mathbb{P}) \geq \gamma^*$. Having shown both directions, we can conclude that $\max_{\mathbb{P}\in\mathbb{P}_2(\mathcal{G})} \Psi(\mathbb{P}) = \gamma^*$ and we have proven the theorem. ■

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