Extended Abstracts of the
25th International Symposium on Mathematical Theory of Networks and Systems
Bayreuth, Germany, 12-16 September 2022

On Internal Stability of Diffusive-Coupling and the Dangers of Cancel Culture ★

Gal Barkai * **Leonid Mirkin** * **Daniel Zelazo** **

* *Faculty of Mechanical Engineering, Technion—IIT, Haifa 3200003, Israel (e-mails: galbarkai@campus.technion.ac.il & mirkin@technion.ac.il)*
** *Faculty of Aerospace Engineering, Technion—IIT, Haifa 3200003, Israel (e-mail: dzelazo@technion.ac.il)*

Abstract: We study internal stability in the context of diffusively-coupled control architecture, common in multi-agent systems (e.g. the celebrated consensus protocol). We derive a condition under which the system can be stabilized by no controller from that class. The condition says effectively that diffusively-coupled controllers cannot stabilize agents that share common unstable dynamics, directions included. This class always contains a group of homogeneous unstable agents, like integrators. We argue that the underlying reason is intrinsic cancellations of unstable agent dynamics by such controllers, even static ones, where directional properties play a key role. The intrinsic lack of internal stability explains the notorious behavior of some distributed control protocols when affected by measurement noise or exogenous disturbances.

Keywords: Multi-agent systems, structural properties, stability.

1. INTRODUCTION

A multi-agent system (MAS) is a collection of independent systems (agents) coupled via pursuit of a common goal. In large-scale MASs the information exchange between agents is normally limited to a subset of the agents, known as *neighbors*. Control laws using only information from neighboring agents are called *distributed*.

This work studies a class of distributed control laws, where only *relative* measurements are exchanged between neighbors. In other words, each agent has access only to the difference between its output and that of each of its neighbours. Relative sensing appears frequently in MAS tasks where absolute measurement are hard to obtain, such as space and aerial exploration and sensor localization, see (Smith and Hadaegh, 2005; Khan et al., 2009; Zelazo and Mesbahi, 2011b) and the references therein. Distributed control laws generated by relative information are called *diffusive*, and systems controlled by such laws are called *diffusively coupled*. Diffusive coupling appear naturally in consensus and synchronization problems (Olfati-Saber et al., 2007; Wieland et al., 2011), making them common in the MAS literature. However, diffusively-coupled systems behave poorly when affected by disturbances and noise. Measurement noise rapidly deteriorates performance (Zelazo and Mesbahi, 2011a, §III.A), and even dynamic controllers can hardly attenuate disturbances (Ding, 2015). To illustrate some of these traits, consider a simple example.

1.1 Motivating example

Reaching agreement between autonomous agents is a fundamental building block in multi-agent coordination (Ren and Beard, 2008). In its simplest form, it concerns a group of integrator agents described by $\dot{x}_i(t) = u_i(t)$, which need to synchronize their states x_i in a distributed manner. Namely, it is required to attain

$$\lim_{t\to\infty}\big(x_i(t) - x_j(t)\big) = 0, \quad \forall i, j. \tag{1}$$

by an appropriate choice of control signals u_i with access only to a states of neighboring agents, denoted by the set $\mathcal{N}_i$. This problem can be solved by the celebrated consensus protocol (Olfati-Saber et al., 2007), which is a diffusive state-feedback

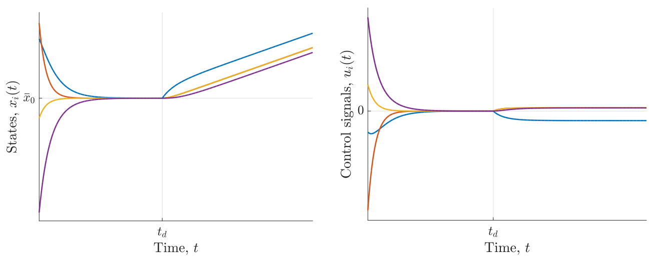
$$u_i(t) = -\sum_{j\in\mathcal{N}_i}\big(x_i(t) - x_j(t)\big). \tag{2}$$

If certain conditions on the communication topology hold, then the control law (2) drives the agents to agreement exponentially fast (Mesbahi and Egerstedt, 2010, Ch. 3).

This is no longer the case if the agent dynamics are affected also by exogenous inputs,

$$\dot{x}_i(t) = u_i(t) + d_i(t) \tag{3}$$

for some independent and unmeasurable load disturbances d_i. Fig. 1 demonstrates what happens with a group of 4 agents controlled by (2) when a unit step disturbance appears at one of them at some time instance $t = t_d$. For $t < t_d$, when the system is undisturbed, the states converge exponentially to the average



(a) states of the agents (b) control signals

Fig. 1. Consensus protocol for agents perturbed at $t = t_d$

★ Supported by the Israel Science Foundation (grants no. 2000/17 and 2285/20).

of their initial conditions and the control signals go to zero. However, for $t > t_d$ the states x_i disagree and diverge, whereas the control signals u_i reach non-zero steady-state values.

The apparent instability of the whole system, manifested in the unboundedness of the states, can be explained by the well-known fact that the consensus protocol has a closed-loop eigenvalues at the origin (Olfati-Saber et al., 2007). Nevertheless, the boundedness of the control signals under such conditions is intriguing. The situation when some signal in the closed-loop system are bounded, whereas some other are not, may indicate unstable *pole-zero cancellations* in the feedback loop (Zhou et al., 1996, Ch. 5.3). However, controller (2) is static and thus has no zeros.

Still, the behavior like that in Fig. 1 prompts a deeper inspection of the *internal stability* property, which is the stability of all possible input/output relations in the system, see (Zhou et al., 1996; Skogestad and Postlethwaite, 2005). To the best of our knowledge, the instability phenomenon above was never explicitly connected with the lack of internal stability or unstable cancellations [1] . This is the starting point of the current study.

1.2 Contribution

In this note we show that the diffusive-coupling distributed control architecture for MASs is intrinsically internally unstable for many common agents configurations. Specifically, we prove that this is the case whenever all agents share common unstable dynamics (directions included for MIMO agents). This, for example, always happens in the group of homogeneous unstable agents, like integrator agents in (3).

We also explain the mechanism for the shown internal instability. It is indeed caused by unstable cancellations in the cascade of the block-diagonal aggregate plant and the diffusively coupled controller. Interesting is that these cancellations are caused not by controller zeros, but rather by an intrinsic spatial deficiency of the diffusive coupling configuration. They are thus independent of particular dynamics in processing relative measurements, only agents dynamics matter. It is worth mentioning that this instability mechanism is unrelated to the decentralized fixed modes (Wang and Davison, 1973).

The internal instability in the form of canceled plant poles explains then observed problems associated with the load disturbance response in some MAS applications.

Notations We extensively use standard notation from algebraic graph theory (Godsil and Royle, 2001). An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite vertex set $\mathcal{V}$ and edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Denote by E the (oriented) incidence matrix of $\mathcal{G}$, defined component-wise by $[E(\mathcal{G})]_{ij} = 1$, when i is the head of edge j, $[E(\mathcal{G})]_{ij} = -1$ when i is the tail of edge j, and 0 otherwise. The matrix $L := EE^\top$ is the combinatorial Laplacian matrix of $\mathcal{G}$. Note that $\mathbb{1} \in \ker E^\top$, thus L has an eigenvalue at the origin with $\mathbb{1}$ as its eigenvector.

The sets of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively, while the notations $\mathbb{C}_0$ and $\bar{\mathbb{C}}_0$ denote the open and closed right half complex plane, respectively. The complex-conjugate transpose of a matrix M is denoted by $M^\top$. The notation $\operatorname{diag}\{M_i\}$ stands for a block-diagonal matrix with diagonal elements M_i. The image (range) and kernel (null) spaces of a matrix M are notated $\operatorname{Im} M$ and $\ker M$, respectively. Given two matrices (vectors) M and N, $M \otimes N$ denotes their Kronecker product. By I_ν, or simply I, we denote the $\nu \times \nu$ identity matrix, by $\mathbb{1}_\nu$, or simply $\mathbb{1}$, the ν dimensional all-ones vector. The notation $\operatorname{spec} G$ refers to the set of eigenvalues if G is a matrix, or the set of poles if $G(s)$ is a proper transfer function. By H_∞ we denote the space of functions holomorphic and bounded in $\mathbb{C}_0$.

[1] Reminiscent reasoning has been mentioned in a formation control problem solved by a diffusive controller in (Fax and Murray, 2004, Sec. III.B), where the a cancelled mode was interpreted as unobservability of absolute motion.

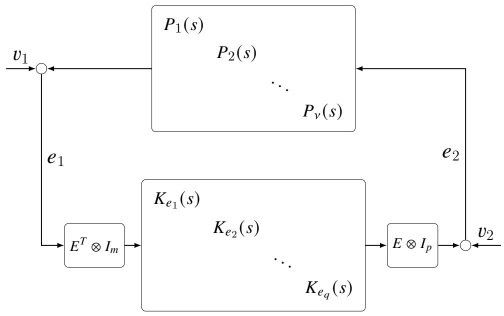


Fig. 2. Block diagram of the closed-loop

2. PROBLEM FORMULATION

Consider ν continuous-time agents P_i, each with m inputs and p outputs. Their aggregate dynamics are denoted as $P := \operatorname{diag}\{P_i\}$, $i = 1, \ldots, \nu$. The interconnection topology of all agents is described by a graph $\mathcal{G}$ with ν nodes and q edges. Agent i and agent j are neighbors in the sense described in Section 1 if they are incident to the same edge. A dynamic controller, $K_e := \operatorname{diag}\{K_{e,j}\}$, $j = 1, \ldots, q$, acts on the relative measurements on the edges. We assume hereafter that all P_i and K_{e_i} are linear time invariant (LTI), finite dimensional [2] , and that their transfer functions are proper.

A general diffusively-coupled architecture can be presented as the interconnection shown in Fig. 2, where the coupling matrix is the incidence matrix of $\mathcal{G}$. This representation is common in passivity-based analysis (Arcak, 2007), and sometimes called the canonical cooperative control structure (Sharf and Zelazo, 2017), (Bullo, 2022, Ch. 9). An equivalent representation can be made using the Laplacian matrix (Bullo, 2022, Ch. 8).

Note that the coupling matrices can be attached to either the plant or the controller, resulting in two distinct problems. One of them considers edge controllers stabilizing diagonal node dynamics (Bürger and De Persis, 2015), while the other a diagonal controller stabilizing the edge dynamics (Zelazo and Mesbahi, 2011a). In this note we consider the former, which includes controller

$$K := (E \otimes I_m) K_e (E^\top \otimes I_p) \tag{4}$$

connected with a diagonal plant, as shown in Fig. 2, where v_1 and v_2 are arbitrary and bounded exogenous signals.

We say that the system in Fig. 2 is internally stable if all four closed-loop transfer functions connecting the exogenous signals v_1 and v_2 with the internal signals e_1 and e_2 are stable, i.e. belong to H_∞. The question studied in this note is under

[2] The arguments below could be extended to infinite-dimensional systems, but the involved technicalities are beyond the scope of this note.

what conditions on the dynamics of the agents P_i there are edge controllers K_{e_j} internally stabilizing this system.

3. THE MAIN RESULT

The main result of this note provides a condition for the interconnection in Fig. 2 to be internally unstable irrespective of the choice of dynamics of K_e. As mentioned, the underlying reason is *cancellations* between poles of P and the controller K. It is well documented that poles cancelled in a cascade between a plant and controller are not truly cancelled (Anderson and Gevers, 1981). They may not appear in one I/O relation, but will still appear in a different one. For SISO systems cancellations are simple to spot and understand, they happen if and only if one system has a pole and the other a zero at the same location. When generalizing this to MIMO, poles and zeros have directional properties, see (Skogestad and Postlethwaite, 2005, §4.6.1) and (Mirkin, 2019, §3.4.2). Thus, directions, and not just locations, have to be considered.

Definition 1. Let G be an LTI system with m inputs and p outputs and (A, B, C, D) be its state space realization with realization poles at $\lambda_i \in \mathbb{C}$. The *input direction* of every λ_i is

$$\mathrm{pdir_i}(G, \lambda_i) := B^\top \ker([\lambda_i I - A]^\top) \subseteq \mathbb{C}^m$$

and its *output direction* is

$$\mathrm{pdir_o}(G, \lambda_i) := C \ker(\lambda_i I - A) \subseteq \mathbb{C}^p.$$

Pole directions span subspaces of either the input or output space. If λ_i is a hidden, i.e. uncontrollable or unobservable, mode of the realization (A, B, C, D), then both $\mathrm{pdir_i}(G, \lambda_i) = \{0\}$ and $\mathrm{pdir_o}(G, \lambda_i) = \{0\}$, which follows by PBH arguments. In the SISO case, $\mathrm{pdir_i}(G, \lambda_i) = \mathrm{pdir_o}(G, \lambda_i) = \mathbb{C}$ whenever λ_i is also a pole of the transfer function $G(s)$, i.e. every pole is excited by every input.

Cancelled poles correspond to unobservable (uncontrollable) modes in either KP or PK (Zhou et al., 1996, Thm. 5.7). The following Lemma provides conditions on pole directions of the agents in the diagram in Fig. 2 under which parts of the dynamics of agents are canceled by the controller K.

Lemma 2. Let P and K be as described in Section 2 and λ be a common pole of all agents P_i.

i) If
$$\bigcap_{i=1}^{\nu} \mathrm{pdir_o}(P_i, \lambda) \neq \{0\},$$
then λ is an unobservable mode of KP.

ii) If
$$\bigcap_{i=1}^{\nu} \mathrm{pdir_i}(P_i, \lambda) \neq \{0\},$$
then λ is an uncontrollable mode of PK.

The cancellation of λ in Lemma 2 is independent of the controller dynamics. In MIMO systems poles of the plant can be canceled not by zeros of the controller, but rather by a normal rank deficiency of the latter. This is exactly what happens in the diffusively-coupled interconnection in Fig. 2. Namely, the intrinsic singularity of the incidence matrix, present in every diffusive controller, might cancel plant poles. A formal condition for that is stated in Lemma 2.

Since cancelled poles remain poles of at least one closed-loop transfer function (Anderson and Gevers, 1981), the above Lemma immediately implies the main result.

Theorem 3. Let P_i, $i = 1, \ldots, \nu$, be LTI finite-dimensional agents with proper transfer functions. If $\lambda \in \bar{\mathbb{C}}_0$ is a pole of each one of them such that

$$\bigcap_{j=1}^{\nu} \mathrm{pdir_i}(P_j, \lambda) \neq \{0\} \tag{5a}$$

or

$$\bigcap_{j=1}^{\nu} \mathrm{pdir_o}(P_j, \lambda) \neq \{0\}, \tag{5b}$$

then the interconnection shown in Fig. 2 is internally unstable irrespective of the choice of K_e. Moreover, if this λ is not a zero of K_e, then condition (5a) implies that λ is the pole of the closed-loop transfer function from v_2 to e_1, while condition (5b) implies that λ is the pole of the closed-loop transfer function from v_1 to e_2.

Theorem 3 asserts that that any common dynamics, determined by poles and corresponding directions, are cancelled by the diffusive coupling. This has an interesting immediate corollary. If the agents are homogeneous they share their entire dynamics, both stable and unstable, thus the diffusive structure can be thought of as cancelling an *entire agent*. This not only proves the unobservability of the mode at the origin claimed in (Fax and Murray, 2004), but proves that every pole loses multiplicity in the cascade.

This may have ramifications not only about the stability of the system, but also of its maximal attainable performance. For example, it explains the observation reported in (Li et al., 2010), where the disturbance rejection performance measure of the entire system is upper bounded by that of a single, uncontrolled agent. It also generalizes the observation from (Zelazo and Mesbahi, 2011a), where it was shown that for integrator agents there is always an unobservable mode parallel to span $\mathbb{1}$. Since this direction is in the null space of the incidence matrix, noise or disturbances effecting this mode cannot be attenuated by a diffusive controller. Similarly, this cancellation explains why the cooperative disturbance rejection scheme of (Ding, 2015) cannot reject load disturbances, but only synchronize to it.

4. CONCLUDING REMARKS

We presented necessary conditions for internal stabilizability of diffusively-coupled LTI systems. In particular, we have shown that, for finite-dimensional agents, common dynamics are cancelled by the diffusive controller. The final conclusion is that in numerous multi-agent problems, one cannot simultaneously achieve a cooperative objective and guarantee internal stability using only relative measurements. Extending these results to time-varying graphs and more general systems are subject to current research.

REFERENCES

Anderson, B. and Gevers, M. (1981). On multivariable pole-zero cancellations and the stability of feedback systems. *IEEE Transactions on Circuits and Systems*, 28(8), 830–833.

Arcak, M. (2007). Passivity as a design tool for group coordination. *IEEE Trans. Automat. Control*, 52(8), 1380–1390.

Bullo, F. (2022). *Lectures on Network Systems*. Kindle Direct Publishing, 1.6 edition. URL `http://motion.me.ucsb.edu/book-lns`.

Bürger, M. and De Persis, C. (2015). Dynamic coupling design for nonlinear output agreement and time-varying flow control. *Automatica*, 51, 210–222.

Ding, Z. (2015). Consensus disturbance rejection with disturbance observers. *IEEE Transactions on Industrial Electronics*, 62(9), 5829–5837.

Fax, J.A. and Murray, R.M. (2004). Information flow and cooperative control of vehicle formations. *IEEE Trans. Automat. Control*, 49(9), 1465–1476.

Godsil, C.D. and Royle, G.F. (2001). *Algebraic Graph Theory*. Springer.

Khan, U.A., Kar, S., and Moura, J.M.F. (2009). Distributed sensor localization in random environments using minimal number of anchor nodes. *IEEE Transactions on Signal Processing*, 57(5), 2000–2016.

Li, Z., Duan, Z., Chen, G., and Huang, L. (2010). Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 57(1), 213–224.

Mesbahi, M. and Egerstedt, M. (2010). *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, Princeton.

Mirkin, L. (2019). Linear Control Systems. course notes, Faculty of Mechanical Eng., Technion—IIT. URL `http://leo.technion.ac.il/Courses/LCS/LCSnotes.pdf`.

Olfati-Saber, R., Fax, A., and Murray, R.M. (2007). Consensus and cooperation in networked multi-agent systems. *Proc. IEEE*, 95(1), 215–233.

Ren, W. and Beard, R.W. (2008). *Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications*. Springer-Verlag, London.

Sharf, M. and Zelazo, D. (2017). A network optimization approach to cooperative control synthesis. *IEEE Control Syst. Lett.*, 1(1), 86–91.

Skogestad, S. and Postlethwaite, I. (2005). *Multivariable Feedback Control: Analysis and Design*. John Wiley & Sons, Chichester, 2nd edition.

Smith, R.S. and Hadaegh, F.Y. (2005). Control of deep-space formation-flying spacecraft; relative sensing and switched information. *Journal of Guidance, Control, and Dynamics*, 28(1), 106–114.

Wang, S.H. and Davison, E. (1973). On the stabilization of decentralized control systems. *IEEE Transactions on Automatic Control*, 18(5), 473–478.

Wieland, P., Sepulchre, R., and Allgöwer, F. (2011). An internal model principle is necessary and sufficient for linear output synchronization. *Automatica*, 47(5), 1068–1074.

Zelazo, D. and Mesbahi, M. (2011a). Edge agreement: Graph-theoretic performance bounds and passivity analysis. *IEEE Transactions on Automatic Control*, 56(3), 544–555.

Zelazo, D. and Mesbahi, M. (2011b). Graph-theoretic analysis and synthesis of relative sensing networks. *IEEE Transactions on Automatic Control*, 56(5), 971–982.

Zhou, K., Doyle, J.C., and Glover, K. (1996). *Robust and Optimal Control*. Prentice-Hall, Englewood Cliffs, NJ.