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An emulation approach to sampled-data synchronization

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***Abstract*— This work presents a novel approach for achieving state synchronization of homogeneous LTI agents to a trajectory generated by a prescribed reference generator under intermittent and asynchronous communication. The proposed protocol involves emulating "ideal" global analog dynamics at each agent to generate the control signal between samples. Each agent transmits the *centroid* state of its local emulator rather than its own state vector, which is used to update the emulators at the receiving end. The paper guarantees synchronization with a prescribed reference generator under mild assumptions on the system's structure, persistency of connectivity, and uniform boundedness of sampling intervals. Additionally, the controller parameters are independent of the sampling interval, allowing it to be designed without any a priori knowledge of the sampling sequence. Lastly, a simplified and scalable implementation whose dimension is independent of the number of agents is also proposed.**

***Index Terms*— Sampled-data systems, network control systems.**

I. Introduction

Over the past two decades, control problems involving autonomous agents interacting via communication networks have grown in significance. A central subclass of these are *agreement problems*, often referred to as either consensus [1], [2] or synchronization [3], [4]. In agreement problems, the agents are tasked with asymptotically converging to a common trajectory without a uniform reference signal. Given the absence of such reference, agents must exchange information to ensure their convergence. However, in practical applications, their communication may be limited in various ways. It is convenient to divide these constraints into two categories: spatial and temporal.

Spatial constraints dictate that agents can only communicate with a specific subset, termed their *neighbors* [5], [6]. This type of constraint is especially prevalent in agreement problems and is often analyzed through a blend of system and graph theoretic methods [7], [8]. Temporal constraints determine when these agents can interact, typically confining them to discrete time instances. As a result, agents must generate their control signal based on intermittently available information from the group.

Multi-agent problems with a singular communication constraint are well-studied. When subject to spatial constraints control laws are often rooted in the consensus protocol [1], employing graph-theoretical constructs such as the *graph Laplacian* to naturally restrict the spatial structure. For homogeneous linear time-invariant (LTI) agents doing this reduces the problem to simply designing a local controller that is robust to certain perturbations, see [6] or [8, Sec. 8.3]. Similarly, temporally constrained control laws with a full spatial structure are also generally well understood, even for intermittent and asynchronous sampling [9]. However, the same cannot be said when attempting to address problems subject to both constraints simultaneously, even when considering only LTI agents, see [10] and the references therein.

In such problems, the common practice is to use a sequential design. First, design a spatially constrained control law assuming continuous (analog) communication and then modify it to conform with the actual sampled communication. Such modifications usually introduce conservatism to the design. For example, zero-order hold (ZOH) discretization enforces sufficiently small and conservative gains even for integrator agents under synchronous and periodic sampling [1]. Other methods such as the input-delay approach, e.g. [11], [12], treat the sampling as a perturbation, making them inherently conservative.

This work argues that the "design, discretize, robustify" approach might introduce unnecessary conservatism, which can be mitigated by leveraging sophisticated hold (D/A) devices and exploiting the problem's different time scales. A staple of lumped sampled-data control, generalized hold functions can enhance performance [13] while also accommodating intricate sampling patterns [14]. Furthermore, while inter-agent communication may be limited and sporadic, it is reasonable to assume that agents are equipped to continuously (or considerably faster) monitor their own state. Building on this, recent research [15] tackled the integrator consensus problem with complex communication constraints by designing a generalized hold function that *emulates* an analog (continuous) consensus protocol between samples. The agents exchanged the *centroids* of these emulations when possible and used the incoming information to update their local emulators.

We aim to extend the results of [15] to a more general scenario. Specifically, a leaderless group of homogeneous finite-dimensional LTI agents, tasked with synchronizing their state to a time-varying trajectory. This scenario poses distinct technical challenges. Unlike the integrator case, the resulting closed-loop cannot be decoupled into a series interconnection of purely discrete and continuous systems. To address these complexities, we leverage the freedom in designing the hold function. Instead of emulating a distributed control law, like the consensus protocol, we mimic a centralized controller derived directly from the objective. This analog component is complemented by a discrete consensus-like

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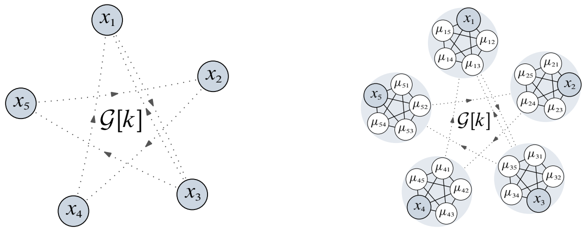


(a) standard (b) emulation-based

Fig. 1. Sampled-data multi-agent control architectures.

update mechanism that incorporates new information from neighbors.

The proposed design approach has several appealing qualities compared to standard sampled-data design methods. First and foremost, the feedback gain can be designed without any prior knowledge of either the sampling sequence or the spatial topology. This is a stark contrast to the previously mentioned approaches where synthesizing the controller gain requires known bounds on the dwell time, sampling intervals, and at times the Laplacian eigenvalues. Moreover, treating the spatial and temporal constraints simultaneously allows us to reduce the problem to a variant of discrete agreement over switching graphs. Hence, agreement is reached under a standard joint connectivity assumption on the induced graphs even for intermittent and asynchronous sampling. Finally, we show that the availability of local information can be leveraged to obtain a reduced order implementation which is independent of the number of agents.

This paper is organized as follows. Section II sets up the problem and communication constraints, followed by Section III which outlines the proposed design approach. The main result is derived in Section IV, §IV-A analyzes the hybrid closed-loop dynamics and proves that they indeed reach asymptotic agreement, while §IV-B proposes a reduced-order and simplified implementation. We end this note with a numerical example in Section V and concluding remarks and future outlook in Section VI.

Notation: The sets of all non-negative integers are denoted as $\mathbb{Z}_+$ and $\mathbb{N}_\nu := \{i \in \mathbb{Z} \mid 1 \le i \le \nu\}$. Sequences with indices from $\mathbb{Z}_+$ are indicated as $\{s_i\}$. The sets of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively, and $\mathbb{C}_0 := \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$. By e_i we understand the ith standard basis vector in $\mathbb{R}^\nu$ and by $\mathbb{1}_\nu$, or simply $\mathbb{1}$ when the dimension is clear from the context, the all-ones vector from $\mathbb{R}^\nu$. The complex-conjugate transpose of a matrix M is denoted by M'. The image (range) and kernel (null space) of a matrix M are notated $\operatorname{Im} M$ and $\ker M$, respectively. The orthogonal projection onto $\operatorname{Im} \mathbb{1}_\nu$ is $P_\mathbb{1} := \mathbb{1}_\nu \mathbb{1}_\nu'/\nu$. Given two matrices (vectors) M and N, $M \otimes N$ denotes their Kronecker product, while $\operatorname{spec}(M)$ refers to the set of all eigenvalues of M.

II. Problem setup

Consider ν homogeneous agents, each with linear dynamics given by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad x_i(0) = x_{i,0}, \quad i \in \mathbb{N}_\nu \tag{1}$$

for some matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, where $x_i(t)$ and $u_i(t)$ are the ith state and control signal respectively. As mentioned in the Introduction, the agents are subject to complex communication constraints, which manifest as restrictions on the information each agent may use to generate $u_i(t)$. To this end, we define $z_i(t)$ as the local information the ith agent may transmit to its neighbors.

The spatial constraints are represented by neighborhood sets, $\mathcal{N}_i(t) \subset \mathbb{N}_\nu \setminus \{i\}$, where each $\mathcal{N}_i(t)$ denotes the neighbors of agent i at time t. This implies that $u_i(t)$ is some function of the local state, $x_i(t)$, and any $z_j(t)$ such that $j \in \mathcal{N}_i(t)$. Mathematically this may be written as $u_i(t) = \kappa(x_i, z_{\mathcal{N}_i(t)})$ where κ is some function and $z_{\mathcal{N}_i(t)}$ represent the neighbors of agent i at time t. The temporal constraints are represented by a strict monotonically increasing of sampling instances $\{s_k\}$, $k \in \mathbb{Z}_+$, where agents may interact only on time instances $t = s_k$. We also use the convention that $t = s_k$ corresponds to the time at the receiving agent. When combined, the two restrict the ith control signal to be of the form

$$u_i(t) = \kappa(x_i(t), z_{\mathcal{N}_i(s_k)}), \ s_k \le t < s_{k+1}. \tag{2}$$

Consequently, at each s_k the collection of neighborhoods $\mathcal{N}_i[k]$ induces a directed graph, $\mathcal{G}[k]$, determining the permitted information exchange. This is illustrated in Fig. 1(a).

Remark 2.1 (Scope of communication constraints): Note that we only assume the existence of the sampling sequence $\{s_k\}$ and not how it is generated. For example, it can be time-triggered, event-triggered, stochastic, or periodic without loss of generality. Second, note that $\{s_k\}$ is a sequence of sampling instances for the entire ensemble, thus $\mathcal{N}_i[k]$ can be empty for certain agents at some $t = s_k$. This allows our framework to encompass asynchronous communication since $\{s_k\}$ is a sequence of all instances on which at least one of the agents received information. $\triangledown$

We consider the following objective in the spirit of [4].

$\boldsymbol{\mathcal{P}_s}$: Given a matrix $A_0 \in \mathbb{R}^{n \times n}$ with $\operatorname{spec}(A_0) \in \mathbb{C} \setminus \mathbb{C}_0$, design control signals $u_i(t)$ satisfying the spatio-temporal constraints that ensures

$$\lim_{t\to\infty} \|x_i(t) - \mathrm{e}^{A_0 t} r_0\| = 0, \quad \forall i \in \mathbb{N}_\nu \tag{3}$$

for some constant $r_0 \in \mathbb{R}^n$ and all initial conditions of agents (1).

It shall be emphasized that the matrix A_0 does not represent a leader node, but rather the shape of required agreement trajectories. Because setting $A_0 = 0$ recovers the consensus problem and setting $A_0 = A$ recovers the classical synchronization [3], $\mathcal{P}_s$ may be viewed as a generalization of both.

III. Proposed architecture

The spatial and temporal constraints completely characterize the permitted information exchange that any permissible controller must respect. When compared with (2), it is clear that simply incorporating a ZOH synchronized with $\{s_k\}$ is just a particular option. Such controllers keep the control signal constant between updates, an additional constraint imposed by the designer rather than by the communication network. We opt for a different solution, one employing a generalized hold function [13] designed for the objective at hand. This is a well-known principle in lumped sampled-data control systems where, in the absence of an optimal solution, common wisdom dictates that the hold should attempt to reconstruct a "good" LTI continuous-time control law [16, §6.1]. In other words, it should locally *emulate* an analog closed loop, in an open-loop fashion, between samples.

To be more precise, assume that $\mathcal{P}_\mathbf{s}$ is solved by some control law $u_i(t) = \kappa(x_i(t), z_{\mathcal{N}_i(t)})$. Instead of holding the signal constant between samples, we wish to implement a variant of this signal, $u_i(t) = \kappa(x_i(t), \mu_i(t))$, where μ_i is an auxiliary signal with $\mu_i(t) \in \mathbb{R}^{\nu n}$ that we term the *emulator* of agent i. Each local emulated group evolves as the ensemble would under the desired analog control law between samples and is updated at sampling instances. In effect, each agent implements a model of the *entire world* around it and uses it to generate the control signal locally. When new measurements are received each agent updates the emulated states, effectively closing the loop only at sampling instances. This logic is illustrated in Fig. 1(b).

The resulting controller has two components that must be designed: i) an analog control law to emulate and ii) a discrete update mechanism to incorporate new information into the emulators. The analog control law is motivated and designed in §III-A, while the update mechanism and overall controller are discussed in §III-B.

A. The analog control law

The paradigm described hitherto served as the guiding principle in [15], where each agent locally emulated the consensus protocol over some agreed-upon spatial topology. Interestingly, the emulated topology could be chosen as the complete graph, i.e. centralized control law, and still result in a distributed controller respecting the communication constraints. This may be attributed to the fact that the emulators are local in nature and the particular structure of the update mechanism. Motivated by this, we shall design an unconstrained control law to emulate via the generalized hold.

To this end, consider the aggregation of the agents

$$\dot{x}(t) = (I_\nu \otimes A)x(t) + (I_\nu \otimes B)u(t), \quad x(0) = x_0, \tag{4}$$

where $x(t)$ and $u(t)$ correspond vectors stacking their local counterparts $x_i(t)$ and $u_i(t)$. To satisfy (3) the agents must track a common trajectory, implying that asymptotically the aggregate state must lie in the *agreement space*, $\operatorname{Im}(\mathbb{1}_\nu \otimes I_n)$. Introducing the orthogonal projection onto the agreement space, $P_\mathbb{1} \otimes I_n$, the aggregate state can be decomposed as

$$x(t) = \delta(t) + (\mathbb{1}_\nu \otimes I_n)\bar{x}(t). \tag{5}$$

where $\delta(t) := \big((I_\nu - P_\mathbb{1}) \otimes I_n\big)x(t)$ is the *disagreement* and $\bar{x}(t) := (1/\nu)(\mathbb{1}_\nu' \otimes I_n)x(t)$ is the *centroid*. This facilitates the decomposition of (3) into two objectives, one for the disagreement

$$\lim_{t\to\infty} \delta(t) = 0, \tag{6a}$$

and one for the centroid

$$\lim_{t\to\infty} \|\bar{x}(t) - e^{A_0 t} r_0\| = 0. \tag{6b}$$

Because $(P_\mathbb{1} \otimes I_n)x = (\mathbb{1}_\nu \otimes I_n)\bar{x}$, the centroid and disagreement are orthogonal and the two objectives are independent, making it natural to propose some $u_\delta(t)$ and $\bar{u}(t)$ to independently satisfy (6). A state-feedback control in this vein would be

$$u(t) = (I_\nu \otimes F_\mathrm{d})\delta(t) + (\mathbb{1}_\nu \otimes \bar{F})\bar{x}(t) \tag{7}$$

for some gains F_d and $\bar{F}$, under which straightforward algebra results in the following independent dynamics for the disagreement and the centroid,

$$\begin{aligned}\dot{\delta}(t) &= (I_\nu \otimes (A + BF_\mathrm{d}))\delta(t)\\ \dot{\bar{x}}(t) &= (A + B\bar{F})\bar{x}(t).\end{aligned}$$

Note that the disagreement dynamics are in fact ν identical and independent copies of the same nth order system, while the centroid dynamics are simply one nth order system. Therefore, (6a) holds iff

$$A_\mathrm{d} := A + BF_\mathrm{d}$$

is Hurwitz and (6b) holds whenever $A_0 = A + B\bar{F}$. Hence, (7) can solve the problem if we assume that

$\mathcal{A}_\mathbf{1}$: the pair (A, B) is stabilizable and there is $\bar{F}$ such that $A_0 = A + B\bar{F}$.

The above is required for $\mathcal{P}_\mathbf{s}$ to be solved by some unconstrained state-feedback, as is the control law to be emulated. It is possible to consider a more general dynamic controller, such as the one in [4], which would result in a different solvability assumption.

B. Sampled-data control law

Let $\mu_i(t) \in \mathbb{R}^{\nu n}$ denote the ith agent's emulation of the *entire* ensemble under control law (7). Accordingly, by $\mu_{ij}(t) \in \mathbb{R}^n$ we identify the ith agent's emulation of the jth agent's state with the convention that $\mu_{ii}(t) = x_i(t)$. As in §III-A we can define the local disagreement of emulator i as

$$\Delta_i := \big((I_\nu - P_\mathbb{1}) \otimes I_n\big)\mu_i$$

and the centroid of the ith emulator as

$$\bar{\mu}_i := \frac{1}{\nu}(\mathbb{1}_\nu' \otimes I_n)\mu_i.$$

Between sampling instances μ_i emulates system (4), as if controlled by (7), and evolves continuously according to

$$\dot{\mu}_i(t) = \big(I_\nu \otimes A_\mathrm{d} + P_\mathbb{1} \otimes (B(\bar{F} - F_\mathrm{d}))\big)\mu_i(t), \tag{8}$$

for some initial conditions $\mu_i(0) = \mu_{i,0}$. Each agent is controlled by a local version of (7) based upon Δ_i and $\bar{\mu}_i$ instead of their analog counterparts, viz.

$$u_i(t) = (e_i' \otimes F_\mathrm{d})\Delta_i(t) + \bar{F}\bar{\mu}_i(t). \tag{9}$$

As previously discussed, (8) and (9) cannot solve $\mathcal{P}_\mathbf{s}$ on their own since each version of (8) evolves independently from the others. Hence, it must be accompanied by some information exchange mechanism that updates the local emulators while satisfying the spatio-temporal constraints.

In §III-A we have seen that the cooperative aspect of $\mathcal{P}_\mathbf{s}$ can be reduced to a requirement on the centroid given by (6b). Motivated by this, we propose an update mechanism based on the local *emulated centroids*. Namely, at sampling instances, each local emulator is updated according to a discrete system given by

$$\mu_{ij}(s_k^+) = \mu_{ij}(s_k) - \alpha_{ij} \sum_{l\in\mathcal{N}_i[k]} \left(\bar{\mu}_i(s_k) - \bar{\mu}_l(s_k)\right), \tag{10}$$

for all $i \neq j$ and some gains $\alpha_{ij} \in \mathbb{R}$. If gains $\alpha_i = \begin{bmatrix}\alpha_{i1} & \cdots & \alpha_{i\nu}\end{bmatrix}'$ are chosen such that $e_i'\alpha_i = 0$ for all $i \in \mathbb{N}_\nu$, then the closed-loop system of agents (4) controlled by (9) which is generated by (8) and (10) is given by

$$\begin{cases} \dot{\mu}(t) = \left(I_\nu \otimes \left(I_\nu \otimes A_\mathrm{d} + P_\mathbb{1} \otimes B(\bar{F} - F_\mathrm{d})\right)\right)\mu(t) \\ \mu(s_k^+) = (A_\mathrm{jmp}[k] \otimes I_n)\mu(s_k), \quad \mu(0) = \mu_0 \end{cases} \tag{11}$$

where

$$A_\mathrm{jmp}[k] = I_{\nu^2} - \frac{1}{\nu}\sum_{i=1}^{\nu}\sum_{l\in\mathcal{N}_i[k]} (e_i(e_i - e_l)') \otimes (\alpha_i \mathbb{1}_\nu'). \tag{12}$$

Note that the agents communicate only through (10) which is spatially distributed, thus the controller respects the spatio-temporal constraints.

Clearly to solve $\mathcal{P}_\mathbf{s}$ we must make some assumption on the information exchange. Indeed, if new information does not persistently arrive at the agents their trajectories will be governed by the non-interacting flow dynamics. To ensure sufficient information flow we assume that

$\mathcal{A}_2$: there is a strictly increasing sub-sequence of sampling indices $\{k_p\}$ such that for all $p \in \mathbb{Z}_+$ (i) the intervals $s_{k_{p+1}} - s_{k_p}$ are uniformly bounded and (ii) $\cup_{k=k_p+1}^{k_{p+1}} \mathcal{G}[k]$ contains a directed rooted tree.

The above is a standard connectivity assumption [2], [5], [17] guaranteeing that, over bounded intervals of time, new information reaches every agent.

Now, with a controller at hand, we are set to show that it solves $\mathcal{P}_\mathbf{s}$.

IV. The Main Result

In §III-A we saw that the disagreement and centroid dynamics were decoupled by (7), which in turn enabled $\mathcal{P}_\mathbf{s}$ to be reduced into two independent problems. Inspired by this, consider the following partition of the stacked emulators

$$\mu(t) = (I_\nu \otimes \mathbb{1}_\nu \otimes I_n)\bar{\mu}(t) + \Delta(t) \in \mathbb{R}^{\nu^2 n}.$$

Now, $\bar{\mu}(t)$ is an $(n\nu) \times 1$ *block vector*, where the ith $n \times 1$ block contains the centroid of the ith emulator. Similarly, $\Delta(t)$ is a block vector where blocks $\Delta_i(t)$ contain the local disagreement vector of emulator i. Recall that $x_i(t) = \mu_{ii}(t)$, hence the aggregate state is given by

$$x(t) = \bar{\mu}(t) + \sum_{i=1}^{\nu} \left((e_i(e_i' \otimes e_i')) \otimes I_n\right)\Delta(t).$$

The above allows us to pose equivalent conditions for the solution of $\mathcal{P}_\mathbf{s}$ in the same vein as those presented in (6). Namely, if the emulator disagreements asymptotically vanish,

$$\lim_{t\to\infty} \Delta(t) = 0, \tag{13a}$$

and the emulator centroids verify

$$\lim_{t\to\infty} \|\bar{\mu}(t) - \mathbb{1}_\nu \otimes \mathrm{e}^{A_0 t} r_0\| = 0, \tag{13b}$$

then $\mathcal{P}_\mathbf{s}$ is satisfied.

A. Centroid-disagreement separation and synchronization

Unlike their analog counterparts, $\bar{\mu}(t)$ and $\Delta(t)$ are coupled through (10). However, for some choices of update gains α_i they take on a simple structure, allowing for a more streamlined analysis.

Lemma 4.1: If $\mathbb{1}_\nu'\alpha_i = 1$ for all $i \in \mathbb{N}_\nu$, then the disagreements dynamics are given by

$$\begin{cases} \dot{\Delta}(t) = (I_{\nu^2} \otimes A_\mathrm{d})\Delta(t), \quad \Delta(0) = \Delta_0 \\ \Delta(s_k^+) = \Delta(s_k) + \left((B_\mathrm{jmp}[k]\mathcal{L}[k]) \otimes I_n\right)\bar{\mu}(s_k), \end{cases} \tag{14a}$$

with

$$B_\mathrm{jmp}[k] := \sum_{i=1}^{\nu} (e_i e_i') \otimes (\mathbb{1}_\nu/\nu - \alpha_i),$$

and $\mathcal{L}[k]$ is the Laplacian matrix associated with $\mathcal{G}[k]$ [7, §2.3.5]. In addition, the centroid dynamics are given by

$$\begin{cases} \dot{\bar{\mu}}(t) = (I_\nu \otimes (A + B\bar{F})\bar{\mu}(t), \qquad \bar{\mu}(0) = \bar{\mu}_0 \\ \bar{\mu}(s_k^+) = ((I_\nu - \dfrac{1}{\nu}\mathcal{L}[k]) \otimes I_n)\bar{\mu}(s_k) \end{cases} \tag{14b}$$

Proof: The flow dynamics mirror the analog case. Note that $A_\mathrm{jmp}[k]$ is defined identically to the jump map in [15, Lemma 2], hence the result follows from applying it to the jump map of (11). ■

There are two immediate consequences of Lemma 4.1: i) the dynamics of $\bar{\mu}(t)$ are autonomous and do not depend on those of $\Delta(t)$ and ii) $\bar{\mu}(t)$ can be thought of as a discrete input affecting $\Delta(t)$ at time instances $t = s_k$. Consequently, finding conditions under which the centroids satisfy (13b) can be done independently of $\Delta(t)$. This is the purpose of the following result.

Lemma 4.2: If $\mathcal{A}_{1,2}$ holds and $\bar{F}$ is chosen such that $A + B\bar{F} = A_0$, then the emulated centroids (14b) satisfy (13b) with $r_0 := \left(\sum_{i=1}^{\nu} q_i \bar{\mu}_{i,0}\right) = (q' \otimes I_n)\bar{\mu}_0$ where q is some constant vector which depends on the sequence of graphs.

Proof: Omitted because of space limitations. ■

Utilizing $\mathcal{A}_1$, the flow map ensures that each $\bar{\mu}_i(t)$ aligns precisely with the intended trajectory shape. However, if the initial conditions differ the trajectories would be different. This cannot be remedied by the non-interacting flows. On the other hand, the jump map mirrors discrete consensus dynamics. Under $\mathcal{A}_2$, this map will asymptotically steer a constant vector to a fixed consensus point within the agreement space. Thus, only the combined flow and jump dynamics under both assumptions guarantee the solution of $\mathcal{P}_s$ for $\bar{\mu}(t)$. Hence, Lemma 4.2 proves that under $\mathcal{A}_{1,2}$ and a proper choice of $\bar{F}$, $\bar{\mu}(t)$ asymptotically satisfy (13b). The final step would be to show the stability of (14a), which can be thought of as an LTI system with $\bar{\mu}$ as an impulsive input. Moreover, the "input matrix" for these impulses includes a, possibly different, graph Laplacian matrix at each k. In particular, for any graph $\mathbb{1}_\nu \in \ker \mathcal{L}[k]$, therefore if $\bar{\mu}(s_k) \in \operatorname{Im} \mathbb{1}_\nu \otimes I_\nu$ then (14a) will contain no jumps. This is a key property in proving the main result, which is stated below.

Theorem 4.3: Consider agents (1) controlled by (9), generated by emulators (11) and update law (10). If $\mathcal{A}_{1,2}$ holds, then control law (9) solves $\mathcal{P}_s$ for all gains F_d and $\bar{F}$ such that $A_d = A + BF_d$ is Hurwitz and $A + B\bar{F} = A_0$ and all emulator update gains α_{ij} such that $\mathbb{1}'_\nu \alpha_i = 1$ for all $i \in \mathbb{N}_\nu$. Moreover, the emulators asymptotically agree and remain bounded if $e^{A_0 t}$ is bounded.

Proof: By assumption, $\mathbb{1}'_\nu \alpha_i = 1$ for all $i \in \nu$, therefore the condition for Lemma 4.1 holds. Define the centroid error $\epsilon(t) := \bar{\mu}(t) - \mathbb{1}_\nu \otimes e^{A_0 t} r_0$ where r_0 is in Lemma 4.2. By assumption $A + B\bar{F} = A_0$, and A_d is Hurwitz and $\mathcal{A}_{1,2}$ holds, thus from Lemma 4.2 we know that $\bar{\mu}(t) \to \mathbb{1}_\nu \otimes e^{A_0 t} r_0$, or equivalently that $\epsilon(t) \to 0$ from every initial condition. Note that $\mathbb{1}_\nu \in \ker \mathcal{L}[k]$ for all k, thus we can rewrite the jump part of (14a) as

$$\Delta(s_k^+) = \Delta(s_k) + \big(B_{\mathrm{jmp}}[k] \otimes I_n\big)\big(\mathcal{L}[k] \otimes I_n\big)\epsilon(s_k).$$

Since $\epsilon(t) \to 0$ and is bounded, the sequence $\{(\mathcal{L}[k] \otimes I_n)\epsilon(s_k)\}$ is bounded and vanishes asymptotically, reducing (14a) to an LTI system with a bounded and asymptotically vanishing input. The stability of LTI systems with bounded and vanishing inputs is independent of the actual input, therefore since A_d is Hurwitz $\Delta(t) \to 0$. Combining with (13) yields

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \bar{\mu}(t) = \mathbb{1}_\nu \otimes e^{A_0 t} r_0,$$

implying that the agents agree. Similarly, taking the limit for $\mu(t)$ yields

$$\lim_{t\to\infty} \mu(t) = \lim_{t\to\infty} (I_\nu \otimes \mathbb{1}_\nu \otimes I_n)\bar{\mu}(t) = \mathbb{1}_{\nu^2} \otimes e^{A_0 t} r_0,$$

therefore the emulators also agree and remain bounded if $e^{A_0 t}$ is bounded. ∎

B. Can we agree to not disagree?

The obvious drawback of emulation-based control architectures is that each agent must locally emulate the entire group, yielding local controllers whose dimension grows linearly with $n\nu$. This may not be feasible for large networks of high-order agents. In an effort to circumvent that, consider a different representation of (9),

$$u_i(t) = F_d \mu_{ii}(t) + (\bar{F} - F_d)\bar{\mu}_i(t), \tag{9$'$}$$

which is obtained by substituting $\Delta_i(t) = \mu_i(t) - (\mathbb{1}_\nu \otimes I_n)\bar{\mu}_i(t)$. Control law (9′) requires two nth order states, the local emulated centroid and local emulated state, hinting that it might be possible to obtain a reduced order implementation.

Corollary 4.4: If $\mathcal{A}_{1,2}$ holds and each agent can continuously measure its own state then the following nth order local controllers

$$\begin{cases} \dot{\bar{\mu}}_i(t) = (A + B\bar{F})\bar{\mu}_i(t), \quad \bar{\mu}_i(0) = \bar{\mu}_{i,0} \\ \bar{\mu}_i(s_k^+) = \bar{\mu}_i(s_k) - \dfrac{1}{\nu} \displaystyle\sum_{l \in N_i[k]} (\bar{\mu}_i(s_k) - \bar{\mu}_l(s_k)) \\ u_i(t) = F_d x_i(t) + (\bar{F} - F_d)\bar{\mu}_i(t) \end{cases} \tag{15}$$

solves $\mathcal{P}_s$ for all gains $F_d, \bar{F}$ such that $A + B\bar{F} = A_0$ and A_d is Hurwitz.

Proof: By definition $\mu_{ii}(t) = x_i(t)$ which is locally available continuously by assumption, substituting $\mu_{ii}(t)$ with $x_i(t)$ in (9′) gives the first equivalence. Since $x_i(t)$ is locally available, to implement the control each agent needs to implement only $\bar{\mu}_i(t)$. By Lemma 4.1 we know that $\bar{\mu}(t)$ is independent of $\Delta(t)$ thus the rest of (15) follows immediately from considering the local version of (14b). ∎

The above implementation is still distributed and adheres to the spatial and temporal constraints, but now each local controller is only of dimension n regardless of the number of agents. This agrees with the intuition behind the analog control law from §III-A: the agents must track the centroid and drive the disagreements to zero.

Remark 4.1 (Disturbance rejection): The logic is reminiscent of a classic servo-regulation problem, where the control law has a stabilizing component acting on the state and a tracking component acting on the reference signal. This raises the natural question of how will this structure behave in the presence of disturbances, something synchronizing systems tend to do poorly [18]. ▽

V. Illustrative Example

To illustrate the proposed sampled-data protocol, consider a simple system comprised of $\nu = 3$ agents with

$$\dot{x}_i(t) = \begin{bmatrix} 4 & 9 \\ 1 & 4 \end{bmatrix} x_i(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u_i(t)$$

trying to synchronize to $A_0 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. In this case

$$\bar{F} = -\begin{bmatrix} 2 & 4 \end{bmatrix} \quad \text{and} \quad F_d = \begin{bmatrix} -34.6 & 39.2 \end{bmatrix}$$

satisfy the requirements of Theorem 4.3.

We further assume that communication between agents is intermittent and asynchronous, meaning that each agent transmits only at a subset of sampling instances. At each sampling instance $\mathcal{G}[k]$ is a union of any nonempty combination of the three graphs in Fig. 2. The system is simulated for time interval $t \in [0, 30]$, the results of which are shown in Fig. 3. The sampling instances, shown by abscissa ticks, are a random variable such that $s_{k+1} - s_k \in 0.3\,\mathbb{N}_6$. Major ticks indicate the sub-sequence of sampling instances $\{k_p\}$ satisfying $\mathcal{A}_2$. The synchronous trajectory as defined in Lemma 4.2 is plotted in lavender.

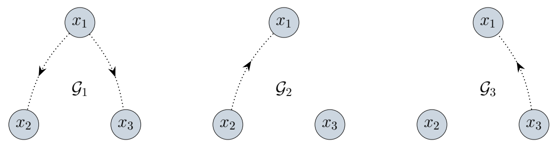
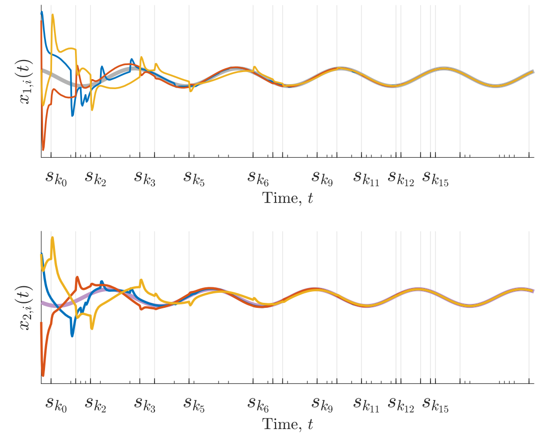


Fig. 2. The three possible graphs.

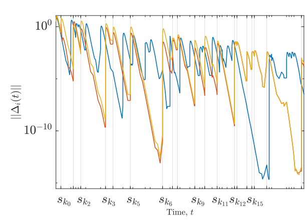
Fig. 3(a) presents the time evolution of the agents states. It can be seen that each component of the state converges to a common trajectory as stated in Theorem 4.3. Fig. 3(b) shows the decay of the emulator disagreement norm, namely $\|\Delta_i(t)\|$, on a logarithmic scale. We can see that $\|\Delta_i(t)\|$ is not monotonically decreasing, which is due to the hybrid nature of the system. The signals $\|\Delta_i(t)\|$ sharply decrease between samples, but might jump up at $t = s_k$, when new information is brought in. Still, there are exponentially decreasing functions upperbounding the combined disagreements norms. The phase portrait of $\bar{\mu}_i(t)$ is given in Fig. 3(c) and displays similar discontinuous behaviour, where each centroid sharply changes its trajectory when the emulators are updated, until they all converge to a common trajectory.

VI. Concluding remarks

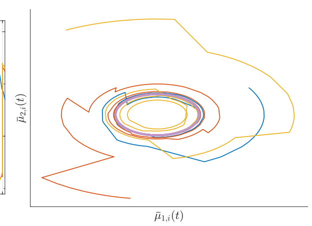
In this note, we addressed time-varying state synchronization of general LTI agents under complex communication constraints. The synchronization is not limited to trajectories generated by the open-loop dynamics, but rather to any dynamics reachable by local state-feedback. We were able to guarantee global asymptotic agreement under mild assumptions on the persistent connectivity of the graphs and sampling instances. Moreover, the control parameters are independent of both the sampling sequence and spatial graphs. These properties are facilitated by a separation between the control law and the information processing mechanism, hinting at possible extensions to more general setups. In particular, extensions to output feedback, disturbance rejection, and systems affected by delays are currently being considered.



(a) Agent states, $x_{1,i}$, $x_{2,i}$, and the synchronous trajectory.



(b) Emulator disagreements (in logarithmic scale).



(c) Phase portrait of $\bar{\mu}_i(t)$.

Fig. 3. The different simulations.

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