A Unified Dissertation on Bearing Rigidity Theory

Giulia Michieletto, *Member, IEEE*, Angelo Cenedese, *Member, IEEE*, and Daniel Zelazo

***Abstract*—This work focuses on bearing rigidity theory, namely, the branch of knowledge investigating the structural properties necessary for multielement systems to preserve the interunit bearings under deformations. The contributions of this work are two-fold. The first one consists in the development of a general framework for the statement of the principal definitions and properties of bearing rigidity. We show that this approach encompasses results existing in the literature, and provides a systematic approach for studying bearing rigidity on any differential manifold in $SE(3)^n$, where n is the number of agents. The second contribution is the derivation of a general form of the rigidity matrix, a central construct in the study of rigidity theory. We provide a necessary and sufficient condition for the infinitesimal rigidity of a bearing framework as a property of the rank of the rigidity matrix. Finally, we present two examples of multiagent systems not encountered in the literature and we study their rigidity properties using the developed methods.**

***Index Terms*—Cyber-physical systems, multiagent systems, networks of autonomous agents, rigidity theory.**

I. Introduction

According to the most general definition, rigidity theory aims at studying the *stiffness* of a given system, understood as a reaction to an induced deformation. The origin of this branch of knowledge dates back to Euler in 1776 [1]. In the centuries since, rigidity analysis has been extended from geometric systems to physical structures, impacting different research areas, ranging from mechanics to biology, and from robotics to chemistry (see [2] and the references therein). An outstanding result in this research thread is the work of Asimow and Roth providing the mathematical description of rigid systems of bars and joints through the notion of a *framework* [3]. This corresponds to the graph-based representation of the system (so that each vertex corresponds to a joint in the structure and each edge represents a bar connecting two elements), jointly with a set of elements in $\mathbb{R}^d$, $d \geq 2$, describing the position of the corresponding units composing the structure [3].

Recently, overcoming the standard bar-and-joints frameworks, rigidity theory has enlarged its focus toward autonomous multiagent systems wherein the connections among the formation elements are virtual and represent sensing relations and capabilities, and/or collective objectives (see, for instance, [4], [5] and the references therein). The concept of a framework has thus been redefined by considering also manifolds more complex than the (n-dimensional) Euclidean space. In these cases, rigidity theory turns out to be an important architectural property of many multiagent systems, where a common global reference frame could be unavailable but the involved devices are characterized by sensing, communication, and movement capabilities. In particular, the rigidity concepts and results suitably fit applications connected to the stabilization and motion control of mobile robot formations and to sensor cooperation for localization, exploration, mapping, and tracking of a target (see, e.g., [6]–[12]).

A. Distance Versus Bearing Rigidity

Within the multiagent systems context, rigidity properties for a given framework deal with agent interactions, either through the available sensing measurements and/or through a common objective function. From this perspective, the literature differentiates between *distance rigidity* and *bearing rigidity*. These two branches of rigidity theory in multiagent systems arose due to the interdependence between the available sensing capabilities in a multiagent system, and the team objective they are trying to solve. For example, target formations specified using interagent distances lead to distributed controllers that require agents to gather relative position measurements [6]. Here, distance rigidity theory emerges in the convergence analysis of these systems. On the other hand, many robotic applications employ direction or bearing-based sensors to achieve coordinated tasks. This motivates a need for objective functions defining formations using interagent bearings, and consequently an extension of rigidity theory to the study of bearing-constrained frameworks [11].

The principal notions about distance rigidity are illustrated in [6], [7], [13]–[19]. These works explain how distance constraints for a framework can be summarized into a properly

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Giulia Michieletto is with the Department of Engineering and Management, University of Padova, 35122 Padova, Italy (e-mail: giulia.michieletto@unipd.it).

Angelo Cenedese is with the Department of Information Engineering, University of Padova, 35122 Padova, Italy (e-mail: angelo.cenedese@unipd.it).

Daniel Zelazo is with the Faculty of Aerospace Engineering, Technion-Israel Institute of Technology, Haifa 32000, Israel (e-mail: dzelazo@technion.ac.il).

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defined matrix whose rank determines the rigidity properties of the system analogously to the case of frameworks embedded in $\mathbb{R}^d$. In such a context, it turns out to be useful to consider the given multiagent system as a bar-and-joint structure, where the agents are modeled as particles (joints) in $\mathbb{R}^d$, and the pairs of interacting devices can be thought as being joined by bars whose lengths enforce the interagent distance constraints.

Bearing rigidity in $\mathbb{R}^2$ (also named *parallel rigidity* [20]) is instead determined by normal constraints over the directions of interacting devices, namely, the edges of the graph associated to the framework, as explained in [21]–[24]. These constraints entail the preservation of the angles formed between pairs of interconnected agents and the lines joining them, i.e., the interagent bearings. Similar interagent direction constraints can be stated to access the rigidity properties of frameworks in $\mathbb{R}^d$ with $d > 2$, where the bearing between two agents coincides with their normalized relative direction vector [10], [11], [25]–[27]. In both cases, the agents are modeled as particles, and the necessary and sufficient condition to guarantee the rigidity properties of a given framework rests upon the rank of a matrix summarizing the involved constraints.

Dealing with a more realistic scenario, in [28]–[31] bearings are assumed to be expressed in the local frame of each agent composing the framework. This implies that each device in the group is modeled as a rigid body having a certain position and orientation with respect to a common global frame which is generally unavailable to the group. In particular, in [28] the attention is focused on multiagent systems acting in the plane, in [29] and [30] the study is extended to the 3D space although limiting the agents attitude kinematics to rotations along only one axis, while in [31] and [32] fully actuated formations are considered by assuming to deal with systems of agents having six controllable degrees of freedom (DOFs). The recent works [33], [34] have also established connections between rigidity and nonlinear observability of dynamic systems: Analogously to the former cases, the rigidity properties of the multiagent systems are studied through the spectral analysis of a matrix that naturally appears in the observability one. However, such an approach abstracts many concepts that are taken into account in this work, as the underlying graph of a framework and the manifold it is embedded in.

B. General Framework and Unified View of Bearing Rigidity

In the past, distance rigidity has been deeply investigated from the theoretical perspective and the related multiagent systems applications are nowadays copious, mainly focusing on formation control and localization (see, e.g., [35]–[38] for a comprehensive overview). Bearing rigidity theory, instead, has been developed only recently, gaining popularity in the last years as a multiagent systems control strategy.

Motivated by the similarities emerging from the existing literature, the first contribution of this work proposes *a unified and general framework for a bearing rigidity theory*. This allows to understand the similarities and differences of the current state-of-the-art results. To this end, Table II in Section V-A provides a comprehensive overview of the principal features of the bearing rigidity theory for frameworks defined on different domains, while in Section VI two examples are given to demonstrate this approach for frameworks that to our knowledge do not appear in the literature. A distinguishing feature of this contribution is the explicit consideration of frameworks over *directed* graphs. Rigidity theory for directed frameworks remains vastly unexplored, with some early results for distance constrained frameworks given in [14], and this work aims to provide a formal foundation for approaching this topic for bearing frameworks. This unified view reveals that all notions of bearing rigidity are related through the so-called rigidity matrix. The second contribution of this work consists of deriving *a general form for the rigidity matrix* that is predictive in the sense that its structure is completely determined by the configuration and interaction graph of the multiagent system. We then provide a necessary and sufficient condition relating the rank of the rigidity matrix to rigidity properties of a given multiagent system, for any agent domain.

The rest of the article is organized as follows. Section II summarizes basic notations and background on graph theory. Section III is devoted to the general definition and properties of bearing rigidity theory. The main results that presented the unified framework for bearing rigidity is given in Section IV. Section V discuses the results in the context of homogeneous formations, and two case studies are given in Section VI. Finally, Section VII is devoted to a brief discussion about colinear formation cases, and Section VIII concludes this article. Appendix, lastly, provides the proof of an auxiliary theoretical result.

II. Preliminaries and Notation

A *graph* is an ordered pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of the vertex set $\mathcal{V} = \{v_1 \dots v_n\}$ and the edge set $\mathcal{E} = \{e_1 \dots e_m\} \subseteq \mathcal{V} \times \mathcal{V}$, having cardinality $|\mathcal{V}| = n$ and $|\mathcal{E}| = m$, respectively. We distinguish between undirected, directed, and oriented graphs. An *undirected graph* is a graph where edges have no orientation; thus, $e_k = (v_i, v_j) \in \mathcal{E}$ is identical to $e_h = (v_j, v_i) \in \mathcal{E}$. Contrarily, a *directed graph* is a graph where edges have orientation so that the edge $e_k = (v_i, v_j) \in \mathcal{E}$ is directed from $v_i \in \mathcal{V}$ (head) to $v_j \in \mathcal{V}$ (tail). An *oriented graph* is an undirected graph jointly with an orientation that is the assignment of a unique direction to each edge, hence only one directed edge ($e_k = (v_i, v_j)$ or $e_h = (v_j, v_i)$) can exist between two vertices $v_i, v_j \in \mathcal{V}$.

For any graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the corresponding *complete graph* $\mathcal{K} = (\mathcal{V}, \mathcal{E}_\mathcal{K})$ is the graph characterized by the same vertex set $\mathcal{V}$, while the edge set consists of all pairs of distinct vertices. Thus, for undirected graphs $|\mathcal{E}_\mathcal{K}| = n(n-1)/2$, and for directed graphs $|\mathcal{E}_\mathcal{K}| = n(n-1)$.

For a directed/oriented graph, the *incidence matrix* $\mathbf{E} \in \mathbb{R}^{n \times m}$ is the $\{0, \pm 1\}$-matrix defined as

$$[\mathbf{E}]_{ik} = \begin{cases} -1, & \text{if } e_k = (v_i, v_j) \in \mathcal{E} \ (\textit{outgoing edge}) \\ 1, & \text{if } e_k = (v_j, v_i) \in \mathcal{E} \ (\textit{ingoing edge}) \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

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and, in a similar way, the matrix $\mathbf{E}_o \in \mathbb{R}^{n\times m}$ is given by

$$[\mathbf{E}_o]_{ik} = \begin{cases} -1, & \text{if } e_k = (v_i, v_j) \in \mathcal{E} \text{ (\textit{outgoing edge})} \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We introduce also the matrices $\bar{\mathbf{E}} = \mathbf{E} \otimes \mathbf{I}_d \in \mathbb{R}^{dn\times dm}$ and $\bar{\mathbf{E}}_\mathrm{o} = \mathbf{E}_o \otimes \mathbf{I}_d \in \mathbb{R}^{dn\times dm}$, where $\otimes$ indicates the Kronecker product, $\mathbf{I}_d$ is the the d-dimensional identity matrix, and $d \geq 2$ refers to the dimension of the considered space.

The d-sphere, i.e., the unit sphere embedded in $\mathbb{R}^{d+1}$, is denoted as $\mathbb{S}^d$. We recall that the 1D manifold $\mathbb{S}^1$ (corresponding to the unit circle) is isomorphic to the special orthogonal group $SO(2) = \{\mathbf{R}\in\mathbb{R}^{2\times 2} \mid \mathbf{R}\mathbf{R}^\top = \mathbf{I}_2,\ \det(\mathbf{R}) = +1\}$ which can be parametrized by a single angle $\alpha \in [0, 2\pi)$. The special orthogonal group $SO(3) = \{\mathbf{R}\in\mathbb{R}^{3\times 3} \mid \mathbf{R}\mathbf{R}^\top = \mathbf{I}_3, \det(\mathbf{R}) = 1\}$, instead, is not isomorphic to the 2-sphere $\mathbb{S}^2$, but it holds that $\mathbb{S}^2 = SO(3)/SO(2)$. In addition, the Cartesian product $\mathbb{R}^d \times SO(d)$ corresponds to the special Euclidean group $SE(d)$.

The vectors of the canonical basis of $\mathbb{R}^d$ are indicated as $\mathbf{e}_i,\ i \in \{1 \dots d\}$, and they have a one in the ith entry and zeros elsewhere. We denote with $\mathbf{0}_d$ and $\mathbf{1}_d$ the vectors in $\mathbb{R}^d$ having all the entries equal to zero and one, respectively, whereas $\mathbf{0}_{d_1\times d_2} \in \mathbb{R}^{d_1\times d_2}$ is the matrix having all the entries equal to zero. Given $\mathbf{x} \in \mathbb{R}^d$, its Euclidean norm is referred as $\|\mathbf{x}\|$. We define the *orthogonal projection operator* as

$$\mathbf{P} : \mathbb{R}^d \to \mathbb{R}^{d\times d}, \quad \mathbf{P}(\mathbf{x}) = \mathbf{I}_d - \frac{\mathbf{x}}{\|\mathbf{x}\|}\frac{\mathbf{x}^\top}{\|\mathbf{x}\|} \tag{3}$$

that maps any (nonzero) vector to its orthogonal complement. Hence, $\mathbf{P}(\mathbf{x})\mathbf{y}$ indicates the projection of $\mathbf{y} \in \mathbb{R}^d$ onto the orthogonal complement of $\mathbf{x} \in \mathbb{R}^d$. Given two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, their cross product is denoted as $\mathbf{x} \times \mathbf{y} = [\mathbf{x}]_\times \mathbf{y} = -[\mathbf{y}]_\times \mathbf{x}$, where the map $[\cdot]_\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ associates each vector $\mathbf{x} \in \mathbb{R}^3$ to the corresponding skew-symmetric matrix belonging to the special orthogonal algebra $\mathfrak{so}(3)$.

Given a matrix $\mathbf{A} \in \mathbb{R}^{p\times q}$, $\mathbf{A}^\top \in \mathbb{R}^{q\times p}$ represents its *transpose* and its *null space* and *image space* are denoted as $\ker(\mathbf{A})$ and $\mathrm{Im}(\mathbf{A})$, respectively. The dimension of $\mathrm{Im}(\mathbf{A})$ is indicated as $\mathrm{rk}(\mathbf{A})$, whereas $\mathrm{null}(\mathbf{A})$ stands for the nullity of the matrix, namely, $\mathrm{null}(\mathbf{A}) = \dim(\ker(\mathbf{A}))$. For the well-known rank–nullity theorem it is $\mathrm{rk}(\mathbf{A}) = q - \mathrm{null}(\mathbf{A})$. In addition, it holds that $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{A}^\top)$. In the rest of the article we use the notation $\mathrm{diag}(\mathbf{A}_k) \in \mathbb{R}^{rp\times rq}$ to indicate the block diagonal matrix associated to the set $\{\mathbf{A}_k \in \mathbb{R}^{p\times q}\}_{k=1}^r$.

Finally, given the function $f : \mathcal{X} \to \mathcal{Y}$ and the sets $\mathcal{A} \subset \mathcal{X}$, and $\mathcal{B} \subset \mathcal{Y}$, then $f(\mathcal{A}) = \{f(x) \in \mathcal{Y} : x \in \mathcal{A}\}$ is called the *image* of $\mathcal{A}$ under f, and $f^{-1}(\mathcal{B}) = \{x \in \mathcal{X} : f(x) \in \mathcal{B}\}$ is called *preimage* of $\mathcal{B}$ under f.

III. Bearing Rigidity: Definitions and Properties

In this section we introduce the main concepts related to bearing rigidity theory.

A. Framework Formation Model

Consider a generic formation of $n \geq 3$ agents, wherein each agent is associated to an element of the differential manifold $\mathcal{D}_i \subseteq SE(3), i \in \{1 \dots n\}$, describing its configuration and motion constraints[1]. In detail, introducing a global frame common (but not necessarily available) to all the agents in the group, the *configuration* $\chi_i \in \mathcal{D}_i$ of the ith agent coincides with its position when modeled as a particle, or with the pair of its position and (partial/full) attitude, i.e., with its (partial/full) pose, when the rigid body model is assumed. We now introduce the notion of a framework.

Definition 1 (Framework in $\bar{\mathcal{D}}$): A framework embedded in the product differential manifold $\bar{\mathcal{D}} = \prod_{i=1}^n \mathcal{D}_i \subseteq SE(3)^n$ is an ordered pair $(\mathcal{G}, \chi)$ consisting of a (directed or undirected) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = n$, and a map $\chi : \mathcal{V} \to \bar{\mathcal{D}}$, such that $v_i \mapsto \chi(v_i) := \chi_i \in \mathcal{D}_i$. We refer to $\chi = (\chi_1 \dots \chi_n) \in \bar{\mathcal{D}}$ as the *formation configuration*.

The framework model characterizes a formation in terms of the agent configuration, where χ can be thought of as an embedding of the graph into $\bar{\mathcal{D}}$, and the agents are associated with nodes in the graph. In the study of bearing rigidity, we are interested in the bearing vector between pairs of agents that are connected by an edge in $\mathcal{G}$. Note that $\mathcal{G}$ can be directed or undirected. In rigidity theory, it is typically assumed that the graph is not time-varying, and we adopt this assumption here.

We introduce also the following definitions on formations.

Definition 2 (Noncolinear Formation): An n-agent formation modeled as a framework $(\mathcal{G}, \chi)$ in $\bar{\mathcal{D}}$ is *noncolinear* if all agents are in distinct positions and do not lie along the same line in the global frame.

Definition 3 (Homogeneous Formation): An n-agent formation modeled as a framework $(\mathcal{G}, \chi)$ in $\bar{\mathcal{D}}$ is *homogeneous* if $\mathcal{D}_i = \mathcal{D}\ \forall i \in \{1 \dots n\}$, hence $\bar{\mathcal{D}} = \mathcal{D}^n$. Otherwise, the formation is *heterogeneous*.

Hereafter, we focus on noncolinear formations, albeit the colinear case is discussed in Section VII. Note that, for a noncolinear homogeneous n-agent formation, the $(n \times d)$ matrix describing the agents position in $\mathbb{R}^d$ is of rank greater than 1.

Although the stated assumptions regard the agents state and motion constraints, for a given formation the bearing rigidity properties are related to the bearing vector between neighboring agents. According to the framework model, every edge $e_k = e_{ij} = (v_i, v_j) \in \mathcal{E}$ ($|\mathcal{E}| = m$) represents a *bearing measurement* $\mathbf{b}_k = \mathbf{b}_{ij}$ defined in the differential manifold $\mathcal{M}_k \subseteq \mathbb{S}^2$ and recovered by the ith agent which is able to sense the jth agent, $i, j \in \{1 \dots n\}, i \neq j$. The *bearing measurements domain* can now be expressed as $\bar{\mathcal{M}} = \prod_{k=1}^m \mathcal{M}_k \subseteq \mathbb{S}^{2\,m}$. For homogeneous formations, it is $\bar{\mathcal{M}} = \mathcal{M}^m$ with $\mathcal{M}_k = \mathcal{M} \forall k \in \{1 \dots m\}$. The available measurements can be expressed in the global frame or in the local frame in-built with each agent/node (and, thus, defined according to $\mathcal{D}_i$). However, in both cases, these are related to the framework configuration according to the following definition, where an arbitrary edge labeling is introduced.

Definition 4 (Bearing Function): Given an n-agent formation modeled as a framework $(\mathcal{G}, \chi)$ in $\bar{\mathcal{D}}$, the *bearing function* is the map $\mathbf{b}_\mathcal{G} : \bar{\mathcal{D}} \to \bar{\mathcal{M}}$ associating the formation configuration

[1]Here, we focus on real world scenarios, nonetheless the definitions and properties provided in the following are valid also for the case $\mathcal{D}_i = \mathbb{R}^d$ with $d > 3$ discussed, e.g., in [26]

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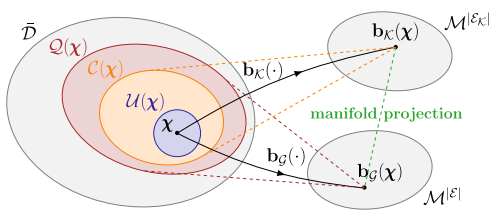

Fig. 1. Graphical interpretation of condition (4).

$\chi \in \bar{\mathcal{D}}$ to the vector $\mathbf{b}_{\mathcal{G}}(\chi) = \begin{bmatrix}\mathbf{b}_1^\top & \dots & \mathbf{b}_m^\top\end{bmatrix}^\top \in \bar{\mathcal{M}}$, stacking all the available bearing measurements.

Observe that the bearing function determines the *shape* of the formation in terms of relative pose among all the agents. One of the central questions in bearing rigidity theory is if a given formation with its bearing function *uniquely* defines the shape. This will explored in the sequel.

Hereafter, the framework model is adopted to refer to an n-agent formation (implying $n \geq 3$) and the two concepts (framework and formation) are assumed to be equivalent.

B. Rigidity Properties of Static Frameworks

Definition 4 allows to introduce the first two notions related to bearing rigidity theory, namely, the equivalence and the congruence of different frameworks.

Definition 5 (Bearing Equivalence): Two frameworks $(\mathcal{G}, \chi)$ and $(\mathcal{G}, \chi')$ are *bearing equivalent (BE)* if $\mathbf{b}_{\mathcal{G}}(\chi) = \mathbf{b}_{\mathcal{G}}(\chi')$.

Definition 6 (Bearing Congruence): Two frameworks $(\mathcal{G}, \chi)$ and $(\mathcal{G}, \chi')$ are *BC* if $\mathbf{b}_{\mathcal{K}}(\chi) = \mathbf{b}_{\mathcal{K}}(\chi')$, where $\mathcal{K}$ is the complete graph associated to $\mathcal{G}$.

Accounting for the preimage under the bearing function, the set $\mathcal{Q}(\chi) = \mathbf{b}_{\mathcal{G}}^{-1}(\mathbf{b}_{\mathcal{G}}(\chi)) \subseteq \bar{\mathcal{D}}$ includes all the formation configurations $\chi' \in \bar{\mathcal{D}}$, such that $(\mathcal{G}, \chi')$ is BE to $(\mathcal{G}, \chi)$, while the set $\mathcal{C}(\chi) = \mathbf{b}_{\mathcal{K}}^{-1}(\mathbf{b}_{\mathcal{K}}(\chi)) \subseteq \bar{\mathcal{D}}$ contains all the formation configurations $\chi' \in \bar{\mathcal{D}}$, such that $(\mathcal{G}, \chi')$ is BC to $(\mathcal{G}, \chi)$. Trivially, it follows that $\mathcal{C}(\chi) \subseteq \mathcal{Q}(\chi)$.

The definition of these sets allows to introduce the (local and global) property of bearing rigidity.

Definition 7 (Bearing Rigidity in $\bar{\mathcal{D}}$): A framework $(\mathcal{G}, \chi)$ is (locally) *bearing rigid (BR)* in $\bar{\mathcal{D}}$ if there exists a neighborhood $\mathcal{U}(\chi) \subseteq \bar{\mathcal{D}}$ of χ, such that

$$\mathcal{Q}(\chi) \cap \mathcal{U}(\chi) = \mathcal{C}(\chi) \cap \mathcal{U}(\chi). \tag{4}$$

Definition 8 (Global Bearing Rigidity in $\bar{\mathcal{D}}$): A framework $(\mathcal{G}, \chi)$ is *globally bearing rigid (GBR)* in $\bar{\mathcal{D}}$ if every framework which is BE to $(\mathcal{G}, \chi)$ is also BC to $(\mathcal{G}, \chi)$, i.e., $\mathcal{Q}(\chi) = \mathcal{C}(\chi)$.

Fig. 1 provides a graphical interpretation of condition (4) highlighting the relation between the sets $\mathcal{Q}(\chi)$, $\mathcal{C}(\chi)$, and $\mathcal{U}(\chi)$. The existence of a neighborhood in the configurations space is not present in Definition 8 of global bearing rigidity. As a consequence, this property results to be stronger than the previous one as proved in the next theorem.

Proposition 1: A GBR framework $(\mathcal{G}, \chi)$ is also BR.

Proof: For a GBR framework $(\mathcal{G}, \chi)$, it holds that $\mathcal{Q}(\chi) = \mathcal{C}(\chi)$. Consequently, condition (4) is valid for $\mathcal{U}(\chi) = \bar{\mathcal{D}}$ demonstrating that the framework is BR. ■

C. Rigidity Properties of Dynamic Frameworks

All the properties previously defined concern rigidity for *static* frameworks. In real-world scenarios, however, agents are generally able to move with or within a formation, and, thus, to change their configuration. Here, the motion constraints are captured by the configuration space $\mathcal{D}_i$ of each agent. For this reason, in this section we assume to deal with *dynamic* agent formations, namely, formations modeled as frameworks $(\mathcal{G}, \chi)$ with fixed sensing graph $\mathcal{G}$ and χ subject to a deformation implied by the variation of the single agent configuration. Formally, we consider the motion of an agent along a curve in its configuration space parameterized by a variable $t \in [0, 1]$. Thus, we have that $\chi_i = \chi_i(t) \in \mathcal{D}_i$ and $\chi = \chi(t) = \{\chi_1(t) \dots \chi_n(t)\} \in \bar{\mathcal{D}}$.

In this context, we introduce the *variation* $\boldsymbol{\delta}_i$ defined in the ith agent *variations domain* $\mathcal{I}_i$, depending on the ith agent motion constraints. In particular, we assume that

$$\frac{d\chi_i(t)}{dt} = f_i(\chi_i(t), \boldsymbol{\delta}_i),$$

where $f_i : \mathcal{D}_i \times \mathcal{I}_i \to \mathcal{D}_i$ is a continuous function. Accounting for the whole formation, we can stack all $\boldsymbol{\delta}_i$ into a vector $\boldsymbol{\delta} \in \bar{\mathcal{I}}$, where $\bar{\mathcal{I}} = \prod_{i=1}^n \mathcal{I}_i$ is the *variations domain*. Note that, for homogeneous formations, we have $\mathcal{I}_i = \mathcal{I}$, and thus, $\bar{\mathcal{I}} = \mathcal{I}^n$. Hereafter, we interpret the variable t as a *time* variable and $\boldsymbol{\delta}$ as the vector of commands acting on the formation to attain the desired evolution from an initial formation configuration $\chi(0)$ to a final one $\chi(1)$.

Given this setup, our aim is to identify the conditions under which a given dynamic formation deforms while maintaining its rigidity features, i.e., preserving the existing bearings among the agents over time. The relation between $\boldsymbol{\delta}$ and the derivative of the bearing function, clarified in the next definition, constitutes the starting point for the study of the formation rigidity properties.

Definition 9 (Bearing Rigidity Matrix): For a given (dynamic) framework $(\mathcal{G}, \chi(t))$, the *bearing rigidity matrix* is the matrix $\mathbf{B}_{\mathcal{G}}(\chi(t))$ that satisfies the relation

$$\dot{\mathbf{b}}_{\mathcal{G}}(\chi(t)) = \frac{d}{dt}\mathbf{b}_{\mathcal{G}}(\chi(t)) = \mathbf{B}_{\mathcal{G}}(\chi(t))\boldsymbol{\delta}. \tag{5}$$

Indeed, $\mathbf{B}_{\mathcal{G}}(\chi(t))$ can be interpreted as a Jacobian matrix within a differential geometry perspective, whose dimensions depend on the spaces $\bar{\mathcal{M}}$ and $\bar{\mathcal{I}}$. Nevertheless, one can observe that the null space of $\mathbf{B}_{\mathcal{G}}(\chi(t))$ always identifies all the (first-order) deformations of $\chi(t)$ that keep the bearing measurements unchanged, following from the Taylor series expansion of the bearing function. From a physical perspective, such variations of $(\mathcal{G}, \chi(t))$ can be interpreted as sets of command inputs to provide to the agents to drive the formation from an initial configuration $\chi(0)$ to a final one $\chi(1)$ belonging to the set $\mathcal{Q}(\chi(0))$ of equivalent formation configurations. Note that the existence of such paths further implies a smooth mapping from $\chi(0)$ to $\chi(1)$, and in this sense, $\boldsymbol{\delta}$ can be interpreted as an instantaneous velocity [39].

Definition 10 (Infinitesimal Variation): For a given (dynamic) framework $(\mathcal{G}, \chi(t))$, a variation $\boldsymbol{\delta} \in \bar{\mathcal{I}}$ is *infinitesimal* if and only if $\boldsymbol{\delta} \in \ker(\mathbf{B}_{\mathcal{G}}(\chi(t)))$.

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It directly results from (5) that an infinitesimal variation preserves the bearing measurements among all interacting agents. For a given $(\mathcal{G}, \chi(t))$, there may be many infinitesimal variations. However, there exist infinitesimal variations that hold for *any* graph. This follows from the next result.

Theorem 1: Given a dynamic framework $(\mathcal{G}, \chi(t))$, and denoting as $\mathcal{K}$ the complete graph associated to $\mathcal{G}$, it holds that $\ker(\mathbf{B}_{\mathcal{K}}(\chi(t))) \subseteq \ker(\mathbf{B}_{\mathcal{G}}(\chi(t)))$.

Proof: Since each edge of the graph $\mathcal{G}$ belongs to the graph $\mathcal{K}$, the equations set $\mathbf{B}_{\mathcal{G}}(\chi(t))\boldsymbol{\delta} = \mathbf{0}_\mu$ constitutes a subset of the equations set $\mathbf{B}_{\mathcal{K}}(\chi(t))\boldsymbol{\delta} = \mathbf{0}_\mu$ with μ depending on $\bar{\mathcal{M}}$. Then $\boldsymbol{\delta} \in \ker(\mathbf{B}_{\mathcal{K}}(\chi(t)))$ implies $\boldsymbol{\delta} \in \ker(\mathbf{B}_{\mathcal{G}}(\chi(t)))$. ■

In light of Theorem 1, we introduce the notion of *trivial variations* by considering the infinitesimal variations related to the complete graph $\mathcal{K}$. These ensure the measurements preservation for each pair of nodes in the formation, i.e., the formation shape preservation in terms of relative poses among the agents.

Definition 11 (Trivial Variation): For a given (dynamic) framework $(\mathcal{G}, \chi(t))$, a variation $\boldsymbol{\delta} \in \bar{\mathcal{I}}$ is *trivial* if and only if $\boldsymbol{\delta} \in \ker(\mathbf{B}_{\mathcal{K}}(\chi(t)))$, where $\mathbf{B}_{\mathcal{K}}(\chi(t))$ is the bearing rigidity matrix computed for the complete graph $\mathcal{K}$ associated to $\mathcal{G}$.

Theorem 1 is fundamental for the next definition that constitutes a key concept in rigidity theory.

Definition 12 (Infinitesimal Bearing Rigidity in $\bar{\mathcal{D}}$): A (dynamic) framework $(\mathcal{G}, \chi(t))$ is *infinitesimally bearing rigid (IBR)* in $\bar{\mathcal{D}}$ if $\ker(\mathbf{B}_{\mathcal{G}}(\chi(t))) = \ker(\mathbf{B}_{\mathcal{K}}(\chi(t)))$. Otherwise, it is *infinitesimally bearing flexible (IBF)*.

From a physical point of view, a framework $(\mathcal{G}, \chi(t))$ is IBR if its *trivial variations set* $\mathcal{S}_t := \ker(\mathbf{B}_{\mathcal{K}}(\chi(t)))$ coincides with its *infinitesimal variations set* $\mathcal{S}_i := \ker(\mathbf{B}_{\mathcal{G}}(\chi(t)))$. A variation in the set $\mathcal{S}_i$ ($\mathcal{S}_t$) induces a deformation of the configuration $\chi(0)$ into $\chi(1)$ that is bearing equivalent (congruent) to the initial one. Thus, for an IBF framework, there exists at least a variation that deforms the configuration $\chi(0)$ to $\chi(1) \in \mathcal{Q}(\chi(0)) \setminus \mathcal{C}(\chi(0))$.

In the rest of the article, we limit our analysis to the dynamic framework case, and whenever it is possible, the time dependency is dropped out to simplify the notation.

IV. Unified Rigidity Theory

Many of the existing works on the bearing rigidity begin their analysis with a statement of the agent configuration space. The rigidity results are then derived, not in generic terms, but explicitly as a function of the chosen configuration space. A consequence of this approach is the need to rederive and redefine rigidity concepts. In this section, we show that stemming from the general setup given in Section III, we are able to unify the study of bearing rigidity that holds for any $\bar{\mathcal{D}}$ inside $SE(3)^n$. The main realization is that any agent can be interpreted as a rigid body acting in 3D space with constraints on its motions (i.e., constrained to move on a differential manifold inside $SE(3)^n$). This approach leads to a general and constructive form for the rigidity matrix and a necessary and sufficient condition relating infinitesimal bearing rigidity to the rank of this matrix.

We consider a formation, where each ith agent, $i \in \{1 \dots n\}$, is associated to a set of bearing vectors related to its neighbors defined by the graph $\mathcal{G}$. An agent can vary its configuration within $\mathcal{D}_i$ consisting of $c_i^t \in \mathbb{N}$ translational DOFs (TDOFs) and $c_i^r \in \mathbb{N}$ rotational DOFs (RDOFs), where $c_i = c_i^t + c_i^r$ is the dimension of the differential manifold $\mathcal{D}_i$. In this work we consider $\mathcal{D}_i$ in $SE(3)$, so c_i^t and c_i^r are limited in $[0, 3]$.

Independently of c_i, each agent in the group can be modeled as a rigid body and associated to a *local reference frame* $\mathscr{F}_i$ whose origin O_i coincides with the agent center of mass. At each time instant $t \geq 0$, it is thus possible to describe the pose of the agent in the 3D space through the pair $(\mathbf{p}_i(t), \mathbf{R}_i(t)) \in SE(3)$, where the vector $\mathbf{p}_i(t) = \begin{bmatrix} p_i^x(t) & p_i^y(t) & p_i^z(t) \end{bmatrix}^\top \in \mathbb{R}^3$ identifies the position of O_i in the global frame $\mathscr{F}_W$ and the matrix $\mathbf{R}_i(t) \in SO(3)$ defines the orientation of $\mathscr{F}_i$ with respect to $\mathscr{F}_W$. In particular, supposing that the (unit) vectors $\mathbf{e}_h \in \mathbb{S}^2$, $h \in \{1, 2, 3\}$ identify the axes of the global frame, we assume that $\mathbf{R}_i(t) = \mathbf{R}(\{\theta_{i,h}(t), \mathbf{e}_h\}_{h=1}^3)$ meaning that $\mathbf{R}_i(t)$ results from the composition of three consecutive rotations, each of them performed around $\mathbf{e}_h$ of an angle $\theta_{i,h}(t)$, according to a suitable sequence.[2] The parameter c_i, on the other hand, is intrinsically related to the dimension of $\mathcal{D}_i$, and χ_i may not necessarily coincide with the whole pair $(\mathbf{p}_i(t), \mathbf{R}_i(t))$. Specifically, when $c_i < 6$, the agent can vary its pose in 3D space only *partially*.

In light of Section III, the described formation can be modeled as a framework in $\bar{\mathcal{D}} \subseteq SE(3)^n$. Under the assumption that agents do not have access to a global frame, $\mathcal{G}$ is a *directed* graph encoding that bearings are inherently expressed in the local frames and are not necessarily reciprocal between pair of agents. Hence, the directed edge $e_k = (v_i, v_j) \in \mathcal{E}$ refers to the bearing of the jth agent obtained by the ith agent. This can be expressed in terms of the relative position and orientation of the agents in $\mathscr{F}_W$, namely

$$\mathbf{b}_k(t) = \mathbf{b}_{ij}(t) = \mathbf{R}_i^\top(t) s_{ij}(t) \mathbf{p}_{ij}(t) = \mathbf{R}_i^\top(t) \bar{\mathbf{p}}_{ij}(t) \tag{6}$$

where $\mathbf{p}_{ij}(t) = \mathbf{p}_j(t) - \mathbf{p}_i(t) \in \mathbb{R}^d$ is the relative position vector, and $s_{ij}(t) = \|\mathbf{p}_{ij}(t)\|^{-1} \in \mathbb{R}$ is the inverse of the relative distance between the ith and jth agent.

To treat in a unified way multiple domains, we can consider the embedding[3] of each $\mathcal{D}_i$ into the $SE(3)$ manifold, thus considering the given formation as a framework $(\mathcal{G}, \chi(t))$ in $SE(3)^n$. Observing (6), we embed $\bar{\mathcal{M}}$ into $\mathbb{S}^{2m}$ and, according to Definition 4, the bearing function can be expressed as

$$\mathbf{b}_{\mathcal{G}}^+(\chi(t)) = \operatorname{diag}(s_{ij}(t)\mathbf{R}_i^\top(t))\bar{\mathbf{E}}^\top \mathbf{p}(t) \in \mathbb{S}^{2m} \tag{7}$$

where $\mathbf{p}(t) = \begin{bmatrix} \mathbf{p}_1^\top(t) \dots \mathbf{p}_n^\top(t) \end{bmatrix}^\top \in \mathbb{R}^{3n}$ stacks all the agent position vectors. Consistently, the variations domain $\bar{\mathcal{I}}$ can be embedded in $\mathbb{R}^{6n}$ and thus, the vector $\boldsymbol{\delta}$ can be substituted by

$$\boldsymbol{\delta}^+ = \begin{bmatrix} \boldsymbol{\delta}_p^\top \boldsymbol{\delta}_a^\top \end{bmatrix}^\top \in \mathbb{R}^{6n} \tag{8}$$

where $\boldsymbol{\delta}_p \in \mathbb{R}^{3n}$ and $\boldsymbol{\delta}_a \in \mathbb{R}^{3n}$ are defined by padding with zeros the corresponding components of $\boldsymbol{\delta}$ related to the possible variations of the agents position and attitude, respectively.

This allows to introduce the following definition.

[2]This reasoning remains valid for any representation of 3D rotations.

[3]Hereafter, a superscript $+$ is used to highlight the vectors defined in the lifted spaces.

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Definition 13 (Extended Bearing Rigidity Matrix): For a given framework $(\mathcal{G}, \chi(t))$ embedded in $SE(3)^n$, the *extended bearing rigidity matrix* is the matrix $\mathbf{B}_{\mathcal{G}}^{+}(\chi(t)) \in \mathbb{R}^{3m\times 6n}$ that satisfies the relation

$$\dot{\mathbf{b}}_{\mathcal{G}}^{+}(\chi(t)) = \frac{d}{dt}\mathbf{b}_{\mathcal{G}}^{+}(\chi(t)) = \mathbf{B}_{\mathcal{G}}^{+}(\chi(t))\boldsymbol{\delta}^{+}. \tag{9}$$

Note that since $\mathbf{b}_{\mathcal{G}}(\chi(t))^{+}$ can be interpreted as the zero-padded version of the vector $\mathbf{b}_{\mathcal{G}}(\chi(t))$ in (6), the relation (9) corresponds to (5) when accounting for the embedding of $\mathcal{D}_i$ in $SE(3)$. The consistency between $\mathbf{B}_{\mathcal{G}}(\chi(t))$ and $\mathbf{B}_{\mathcal{G}}(\chi(t))^{+}$ is guaranteed by the emergence of zero rows in the latter. Along the same line, we also observe that (8) induces a partition of $\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))$ into two blocks, distinguishing between the bearing variations due to the variations of the agents position and orientation, as formalized in the next proposition.

Proposition 2: The extended bearing rigidity matrix in Definition 13 can be expressed as

$$\mathbf{B}_{\mathcal{G}}^{+}(\chi(t)) = \begin{bmatrix}\mathbf{D}_p(t)\mathbf{U}\bar{\mathbf{E}}^{\top} & \mathbf{D}_a(t)\mathbf{V}\bar{\mathbf{E}}_o^{\top}\end{bmatrix} \in \mathbb{R}^{3m\times 6n} \tag{10}$$

where

1) $\mathbf{D}_p(t), \mathbf{D}_a(t) \in \mathbb{R}^{3m\times 3m}$ are derived from the orthogonal projections of relative position and attitude

$$\mathbf{D}_p(t) = \mathrm{diag}(s_{ij}(t)\mathbf{R}_i^{\top}(t)\mathbf{P}\left(\bar{\mathbf{p}}_{ij}(t)\right)) \tag{11}$$

$$\mathbf{D}_a(t) = -\mathrm{diag}(\mathbf{R}_i^{\top}(t)\left[\bar{\mathbf{p}}_{ij}(t)\right]_{\times}) \tag{12}$$

2) $\mathbf{U} = \mathrm{diag}(\mathbf{U}_{ij}) \in \mathbb{R}^{3m\times 3m}$ and $\mathbf{V} = \mathrm{diag}(\mathbf{V}_{ij}) \in \mathbb{R}^{3m\times 3m}$ take into account the (time-invariant) matrices $\mathbf{U}_{ij}, \mathbf{V}_{ij} \in \mathbb{R}^{3\times 3}$ defining, respectively, the translational directions of the bearing measurement $\mathbf{b}_{ij}$ and ith and jth agents rotation directions in the 3D space with respect to the ith agent frame;
3) $\bar{\mathbf{E}}, \bar{\mathbf{E}}_{\mathrm{o}} \in \mathbb{R}^{3n\times 3m}$ are derived from the (time-invariant) incidence matrix of the graph $\mathcal{G}$.

Proof: According to Definition 13, the extended bearing rigidity matrix $\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))$ has to map the vector $\boldsymbol{\delta}^{+} \in \mathbb{R}^{6n}$ to the derivative of the bearing function. By applying the product rule to (7), it follows that:

$$\begin{aligned}\dot{\mathbf{b}}_{\mathcal{G}}^{+}(\chi(t)) = &\left(\frac{d}{dt}\mathrm{diag}(s_{ij}(t))\right)\mathrm{diag}(\mathbf{R}_i^{\top}(t))\bar{\mathbf{E}}^{\top}\mathbf{p}(t)\\ &+ \mathrm{diag}(s_{ij}(t))\left(\frac{d}{dt}\mathrm{diag}(\mathbf{R}_i^{\top}(t))\right)\bar{\mathbf{E}}^{\top}\mathbf{p}(t)\\ &+ \mathrm{diag}(s_{ij}(t))\mathrm{diag}(\mathbf{R}_i^{\top}(t))\bar{\mathbf{E}}^{\top}\frac{d}{dt}\mathbf{p}(t).\end{aligned} \tag{13}$$

In light of (8), it is possible to distinguish the measurements variations induced by the variations of the agents position $\boldsymbol{\delta}_p$ or attitude $\boldsymbol{\delta}_a$: The former contribution refers to the first and third addendum in (13), while the latter contribution is related to the second one. Accounting for the agents dynamics when embedded in $SE(3)$, equation (13) can then be rearranged as

$$\dot{\mathbf{b}}_{\mathcal{G}}^{+}(\chi(t)) = \mathbf{D}_p(t)\mathbf{U}\bar{\mathbf{E}}^{\top}\boldsymbol{\delta}_p + \mathbf{D}_a(t)\mathbf{V}\bar{\mathbf{E}}_o^{\top}\boldsymbol{\delta}_a \tag{14}$$

where $\mathbf{U}\bar{\mathbf{E}}^{\top}, \mathbf{V}\bar{\mathbf{E}}_o^{\top} \in \mathbb{R}^{3m\times 3n}$ link the bearing measurements variations to the agents variations. In detail, the first addendum in (14) includes the product of three matrices: The matrix $\bar{\mathbf{E}}^{\top}$ translates the variations of the agents' position into relative positions (which are scaled bearings), the block matrix $\mathbf{U}$ embeds the variation of each bearing in the manifold $\mathbb{S}^2$ and, finally, the block matrix $\mathbf{D}_p(t)$ accounts for the projection of each bearing $\mathbf{b}_{ij}$ in the ith agent local frame. An analogous reasoning can be carried out for the second addendum in (14), where $\mathbf{D}_a(t)$ acts similarly to $\mathbf{D}_p(t)$ since the skew matrix of the relative position vector is an orthogonal matrix. ■

For any framework in $\bar{\mathcal{D}} \subseteq SE(3)^n$, Proposition 2, thus, provides a construction method and a general structure for the bearing rigidity matrix that can be decomposed into a part related to the formation configuration ($\mathbf{D}_p(t)$ and $\mathbf{D}_a(t)$), and a part related to the graph ($\bar{\mathbf{E}}$ and $\bar{\mathbf{E}}_{\mathrm{o}}$). While not the focus of this work, this structure may also assist in understanding the combinatorial interpretation of certain rigidity properties. We then observe that the structure of $\mathbf{U}$ and $\mathbf{V}$ induces a level of sparsity in the extended bearing matrix, which may result in the presence of some null columns and null rows.

The embedding of $\bar{\mathcal{D}}$ in $SE(3)^n$ implies two facts. On one hand, both the infinitesimal and the trivial variations sets can be lifted from $\bar{\mathcal{I}}$ into $\mathbb{R}^{6n}$, thus considering the corresponding sets $\mathcal{S}_i^{+}$ and $\mathcal{S}_t^{+}$, such that $|\mathcal{S}_i^{+}| = |\mathcal{S}_i| = q_i$ and $|\mathcal{S}_t^{+}| = |\mathcal{S}_t| = q_t$. These new sets are related to the null space of the matrices $\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))$ and $\mathbf{B}_{\mathcal{K}}^{+}(\chi(t))$. Indeed, we first observe that $\mathcal{S}_i^{+} \subset \ker\left(\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))\right)$ and $\mathcal{S}_t^{+} \subset \ker\left(\mathbf{B}_{\mathcal{K}}^{+}(\chi(t))\right)$ since the extended bearing rigidity matrix can differ from the rigidity matrix because of the emergence of zero columns in correspondence of zero entries of the vector $\boldsymbol{\delta}^{+}$. This fact also justifies the presence of other vectors in the null space of both $\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))$ and $\mathbf{B}_{\mathcal{K}}^{+}(\chi(t))$: These are characterized by zero and nonzero entries in correspondence to nonzero and zero entries of $\boldsymbol{\delta}^{+}$. In addition, because $\mathbf{B}_{\mathcal{K}}^{+}(\chi(t))$ includes additional rows as compared to $\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))$ while the structure of the two matrices is the same in terms of zero columns, we conclude that $\ker\left(\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))\right) = \mathcal{S}_i^{+} \oplus \mathcal{S}_v^{+}$ and $\ker\left(\mathbf{B}_{\mathcal{K}}^{+}(\chi(t))\right) = \mathcal{S}_t^{+} \oplus \mathcal{S}_v^{+}$. The set $\mathcal{S}_v^{+}$, whose cardinality $q_v = |\mathcal{S}_v^{+}|$ corresponds to the number of null columns in $\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))$, represents the set of the *virtual variations* of the evaluated formation and it includes the command inputs inducing variations of the agent configuration that do not affect the bearing measurements but that are also not allowed by the physical constraints on the agents dynamics. On the other hand, since the measurements domain $\bar{\mathcal{M}}$ is also lifted into $\mathbb{S}^{2m}$, q_m null rows may characterize $\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))$ in correspondence to the the zero entries added to the vector $\mathbf{b}_{\mathcal{G}}(\chi(t))$ in order to derive $\mathbf{b}_{\mathcal{G}}(\chi(t))$. We finally provide a rank condition on the extended bearing rigidity matrix that guarantees the infinitesimal rigidity of the corresponding framework embedded in any $\bar{\mathcal{D}}$.

Theorem 2: A noncolinear n-agent formation modeled as a framework $(\mathcal{G}, \chi(t))$ in an arbitrary differential manifold $\bar{\mathcal{D}}$ is IBR if and only if $\mathrm{rk}(\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))) = \mathrm{rk}(\mathbf{B}_{\mathcal{K}}^{+}(\chi(t)))$.

Proof: Because $\mathrm{rk}(\mathbf{B}_{\mathcal{G}}^{+}) = 6n - q_v - q_i$ and $\mathrm{rk}(\mathbf{B}_{\mathcal{K}}^{+}) = 6n - q_v - q_t$, it holds that $\mathrm{rk}(\mathbf{B}_{\mathcal{G}}^{+}(\chi(t))) = \mathrm{rk}(\mathbf{B}_{\mathcal{K}}^{+}(\chi(t)))$ if and only if $q_i = q_t$. Due to Theorem 1, the last equivalence is guaranteed if and only if $\ker(\mathbf{B}_{\mathcal{G}}(\chi(t))) = \ker(\mathbf{B}_{\mathcal{K}}(\chi(t)))$, i.e., the framework is IBR. ■

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TABLE I
PARTICULARIZATION OF THE STRUCTURE OF THE EXTENDED BEARING MATRIX IN (10) FOR THE DIFFERENTIAL MANIFOLDS CONSIDERED IN SECTION V-A

$\mathcal{D}$	$\mathbf{p}_i$	$\mathbf{R}_i$	$\mathbf{U}_{ij}$	$\mathbf{V}_{ij}$
$SE(3)$	$[p_i^x \quad p_i^y \quad p_i^z]^\top$	$\mathbf{R}\left(\{\theta_{i,h}(t), \mathbf{e}_h\}_{h=1}^3\right)$	$\mathbf{I}_3$	$\mathbf{I}_3$
$\mathbb{R}^3 \times \mathbb{S}^1$	$[p_i^x \quad p_i^y \quad p_i^z]^\top$	$\mathbf{R}\left(\theta_i(t), \mathbf{v}\right), \mathbf{v} = \sum_{h=1}^{3} v_h \mathbf{e}_h$	$\mathbf{I}_3$	$[\mathbf{0}_{3\times 2} \ \mathbf{v}]$
$\mathbb{R}^2 \times \mathbb{S}^1$	$[p_i^x \quad p_i^y \quad 0]^\top$	$\mathbf{R}\left(\theta_i(t), \mathbf{e}_3\right)$	$[\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{0}_3]$	$[\mathbf{0}_{3\times 2} \ \mathbf{e}_3]$
$\mathbb{R}^3$	$[p_i^x \quad p_i^y \quad p_i^z]^\top$	$\mathbf{I}_3$	$\mathbf{I}_3$	$\mathbf{0}_{3\times 3}$
$\mathbb{R}^2$	$[p_i^x \quad p_i^y \quad 0]^\top$	$\mathbf{I}_3$	$[\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{0}_3]$	$\mathbf{0}_{3\times 3}$

V. INSIGHT INTO HOMOGENEOUS FORMATIONS

In this section, we deal with homogeneous n-agent formations as described in Definition 3, further assuming that all the agents are able to translate and/or rotate around the *same* directions in the global frame $\mathscr{F}_W$. In this case, we have that $c_i^t = c^t$ and $c_i^r = c^r$, hence $c_i = c \forall i \in \{1 \dots n\}$; furthermore, $c_{\text{tot}} = \sum_{i=1}^n c_i = cn$ represents the total DOFs of the formation corresponding to the dimension of the variation domain $\bar{\mathcal{I}}$. We first focus on bearing rigidity theory for frameworks embedded in different differential manifolds according to the existing literature and we show that expression (10) fits for all the considered cases (Section V-A). Along this line, Table I specifies the variables introduced in Section IV for the considered manifolds. Then, we take into account the IBR condition of Theorem 2 and we discuss the relations between the properties of bearing rigidity, global bearing rigidity, and infinitesimal bearing rigidity (Section V-B).

A. Manifolds Realizations

We recast the results from the literature about bearing rigidity for various domains with the common notation proposed in Section III. Table II summarizes the results presented here, anticipating also the IBR condition discussed in the sequel. Time dependency is dropped out for easing the readability.

1) Bearing Rigidity Theory in $\mathbb{R}^d$: When $\mathcal{D} = \mathbb{R}^d$, $d \in \{2, 3\}$, the attention is focused on formations of n agents wherein each element is modeled as a particle and its configuration coincides with its position $\mathbf{p}_i \in \mathbb{R}^d, i \in \{1 \dots n\}$, in the global frame $\mathscr{F}_W$ that is assumed to be known by all the agents. Each element of the group is, thus, characterized by $c^t = d \in \{2, 3\}$ TDOFs and $c^r = 0$ RDOFs. Such frameworks, studied in [26], represent a suitable model, for example, for teams of mobile sensors interacting in a certain (two-dimensional or three-dimensional) area of interest. Here, the framework model $(\mathcal{G}, \chi)$, when $\bar{\mathcal{D}} = \mathbb{R}^{dn}$, the formation configuration χ is associated to the *positions vector* $\mathbf{p} = \left[\mathbf{p}_1^\top \dots \mathbf{p}_n^\top\right]^\top \in \mathbb{R}^{dn}$, and the graph $\mathcal{G}$ is *undirected* since the particle choice allows to assume bidirectional agent iterations, meaning that neighboring agents are able to reciprocally recover bearing measurements. Following the results from [26], the bearing rigidity matrix can be expressed as

$$\mathbf{B}_{\mathcal{G}}(\chi) = \text{diag}(s_{ij}\mathbf{P}(\bar{\mathbf{p}}_{ij}))\bar{\mathbf{E}}^\top \in \mathbb{R}^{dm \times dn}. \tag{15}$$

Furthermore, given a (noncolinear) n-agent formation modeled as a framework $(\mathcal{G}, \chi)$ in $\mathbb{R}^{dn}$, it is possible to prove (see [26, Lemma 4] and Theorem 5 in Appendix) that its trivial variation set coincides with the $(d+1)$-dimensional set

$$\mathcal{S}_t = \text{span}\{\mathbf{1}_n \otimes \mathbf{I}_d, \mathbf{p}\} \tag{16}$$

describing the translation and uniform scaling of χ.

Accounting for Section IV, we observe that for $\bar{\mathcal{D}} = \mathbb{R}^{dn}$, independently on $d \in \{2, 3\}$, each agent in the group can be modeled as a rigid body having fixed attitude. From a mathematical perspective, this means that $\mathbf{R}_i = \mathbf{I}_3 \forall i \in \{1 \dots n\}$, and that $\mathbf{V}_{ij} = \mathbf{0}_{3\times 3}$ for every bearing measurement both for $d = 2$ and $d = 3$. On the other hand, in correspondence to all the edges of $\mathcal{G}$, it trivially holds that $\mathbf{U}_{ij} = \left[\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{0}_3\right]$ when $d = 2$ (embedding $\mathbb{R}^2$ into $\mathbb{R}^3$) and $\mathbf{U}_{ij} = \mathbf{I}_3$ when $d = 3$. Hence, $\mathbf{D}_a$ in (10) results to be a null matrix, while $\mathbf{D}_p$ corresponds to the bearing rigidity matrix (15). Analyzing then the null space of $\mathbf{B}_{\mathcal{G}}^+(\chi)$ as in (10), the virtual trivial variations result to be $q_v = 4n$ when $d = 2$ and $q_v = 3n$ when $d = 3$ and these corresponds to the three rotational movements, in addition to the translation along the z-axis of the 3D global frame for the $d = 2$ case.

2) Bearing Rigidity Theory in $\mathbb{R}^d \times \mathbb{S}^1$: In a formation wherein each agent configuration is defined in $\mathbb{R}^d \times \mathbb{S}^1$, $d \in \{2, 3\}$, the n components are all characterized by $c^t = d$ TDOFs and $c^r = 1$ RDOF controllable in a decoupled way. This is, for instance, the case of teams of unicycle-modeled ground robots (d=2) or of standard underactuated quadrotors (d=3) whose controllable variables are the position and the yaw angle. The described formation can be modeled as a framework $(\mathcal{G}, \chi)$ in $\bar{\mathcal{D}} = (\mathbb{R}^d \times \mathbb{S}^1)^n$. In this case, χ is associated to both the *positions vector* $\mathbf{p} = \left[\mathbf{p}_1^\top \dots \mathbf{p}_n^\top\right]^\top \in \mathbb{R}^{dn}$ and the *attitudes vector* $\boldsymbol{\theta} = \left[\theta_1 \dots \theta_n\right]^\top \in [0, 2\pi)^n$, while the graph $\mathcal{G}$ is *directed*, since we assume that agents do not have access to the global frame. Basing on [28], the bearing rigidity matrix can be written as

$$\mathbf{B}_{\mathcal{G}}(\chi) = \left[\check{\mathbf{D}}_p \bar{\mathbf{E}}^\top \quad \check{\mathbf{D}}_a \mathbf{E}_o^\top\right] \in {}^{dm \times (d+1)n} \tag{17}$$

where $\bar{\mathbf{E}} \in \mathbb{R}^{dn \times dm}$, $\mathbf{E}_o \in \mathbb{R}^{n \times m}$ are derived from $\mathcal{G}$ and

$$\check{\mathbf{D}}_p = \text{diag}(s_{ij}\mathbf{R}_i^\top \mathbf{P}(\bar{\mathbf{p}}_{ij})) \in \mathbb{R}^{dm \times dm} \quad \text{if } d \in \{2, 3\},$$
$$\check{\mathbf{D}}_a = \begin{cases} -\text{diag}(\mathbf{R}_i^\top \bar{\mathbf{p}}_{ij}^\perp) \in \mathbb{R}^{dm \times m} & \text{if } d = 2 \\ -\text{diag}(\mathbf{R}_i^\top [\bar{\mathbf{p}}_{ij}]_\times \mathbf{v}) \in \mathbb{R}^{dm \times m} & \text{if } d = 3. \end{cases} \tag{18}$$

In (18), $\bar{\mathbf{p}}_{ij}^\perp = \mathbf{R}(\frac{\pi}{2})\bar{\mathbf{p}}_{ij} \in \mathbb{R}^2$, $\mathbf{R}(\frac{\pi}{2}) \in SO(2)$ is the (unit) vector perpendicular to $\bar{\mathbf{p}}_{ij}$ on the plane, while $\mathbf{R}_i \in SO(d)$ identifies the orientation of $\mathscr{F}_i$ with respect to $\mathscr{F}_W$. For a formation on a plane ($d = 2$), the orientation of each agent is (completely) specified by an angle that is univocally associated to a rotation matrix $\mathbf{R}_i = \mathbf{R}(\theta_i) \in SO(2)$, whereas for the 3D case ($d = 3$), $\mathbf{R}_i = \mathbf{R}(\theta_i, \mathbf{v}) \in SO(3)$ denotes the rotation of angle $\theta_i \in [0, 2\pi)$ around the (unit) vector $\mathbf{v} = \sum_{h=1}^3 v_h \mathbf{e}_h$ with $v_h \in \mathbb{R}$, identifying a fixed direction in $\mathscr{F}_W$.

The trivial variations coincide with the translation and uniform scaling of the entire configuration, jointly with the *coordinated rotation*, namely, the equal rotation of all the agents jointly with the equal rotation of the whole formation around its center. The coordinated rotation subspace $\mathcal{R}_{\circlearrowleft}$ is formally determined

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TABLE II
SUMMARY OF THE PRINCIPAL NOTIONS RELATED TO BEARING RIGIDITY THEORY FOR THE DIFFERENTIAL MANIFOLDS CONSIDERED IN SECTION V-A

i-th agent properties				n-agent formation properties		
$\mathcal{D}$	χ_i	$\mathcal{M}$	$\mathbf{b}_{ij}$	$\mathcal{I}$	$\boldsymbol{\delta}$	IBR condition
$\mathbb{R}^d \quad d \in \{2,3\}$	$\mathbf{p}_i \in \mathbb{R}^d$ (d tdofs + 0 rdofs)	$\mathbb{S}^{d-1}$	$\bar{\mathbf{p}}_{ij}$	$\mathbb{R}^{dn}$	$\boldsymbol{\delta}_p = [\dot{\mathbf{p}}_1^\top \quad \dots \quad \dot{\mathbf{p}}_n^\top]^\top$	$\mathrm{rk}\left(\mathbf{B}_{\mathcal{K}}(\chi)\right) = dn - d - 1$
$\mathbb{R}^2 \times \mathbb{S}^1$	$\mathbf{p}_i \in \mathbb{R}^2, \theta_i \in [0, 2\pi)$ (2 tdofs + 1 rdofs)	$\mathbb{S}^1$	$\mathbf{R}_i^\top \bar{\mathbf{p}}_{ij}$	$\mathbb{R}^{3n}$	$\boldsymbol{\delta} = [\boldsymbol{\delta}_p^\top \quad \boldsymbol{\delta}_a]^\top$ $\quad \boldsymbol{\delta}_p = [\dot{\mathbf{p}}_1^\top \dots \dot{\mathbf{p}}_n^\top]^\top$ $\boldsymbol{\delta}_a = [\dot{\theta}_1^\top \dots \dot{\theta}_n^\top]^\top$	$\mathrm{rk}\left(\mathbf{B}_{\mathcal{K}}(\chi)\right) = 3n - 4$
$\mathbb{R}^3 \times \mathbb{S}^1$	$\mathbf{p}_i \in \mathbb{R}^3, \theta_i \in [0, 2\pi)$ (3 tdofs + 1 rdofs)	$\mathbb{S}^2$	$\mathbf{R}_i^\top \bar{\mathbf{p}}_{ij}$	$\mathbb{R}^{4n}$	$\boldsymbol{\delta} = [\boldsymbol{\delta}_p^\top \quad \boldsymbol{\delta}_a]^\top$ $\quad \boldsymbol{\delta}_p = [\dot{\mathbf{p}}_1^\top \dots \dot{\mathbf{p}}_n^\top]^\top$ $\boldsymbol{\delta}_a = [\dot{\theta}_1^\top \dots \dot{\theta}_n^\top]^\top$	$\mathrm{rk}\left(\mathbf{B}_{\mathcal{K}}(\chi)\right) = 4n - 5$
$\mathbb{R}^3 \times SO(3)$	$\mathbf{p}_i \in \mathbb{R}^3, \mathbf{R}_i \in SO(3)$ (3 tdofs + 3 rdofs)	$\mathbb{S}^2$	$\mathbf{R}_i^\top \bar{\mathbf{p}}_{ij}$	$\mathbb{R}^{6n}$	$\boldsymbol{\delta} = [\boldsymbol{\delta}_p^\top \quad \boldsymbol{\delta}_a]^\top$ $\quad \boldsymbol{\delta}_p = [\dot{\mathbf{p}}_1^\top \dots \dot{\mathbf{p}}_n^\top]^\top$ $\boldsymbol{\delta}_a = [\boldsymbol{\omega}_1^\top \dots \boldsymbol{\omega}_n^\top]^\top$	$\mathrm{rk}\left(\mathbf{B}_{\mathcal{K}}(\chi)\right) = 6n - 7$

as

$$\mathcal{R}_{\circlearrowleft} = \begin{cases} \mathrm{span}\left\{\begin{bmatrix} (\mathbf{I}_n \otimes \mathbf{R}(\pi/2))\,\mathbf{p} \\ \mathbf{1}_n \end{bmatrix}\right\}, & \text{if } d = 2 \\ \mathrm{span}\left\{\begin{bmatrix} (\mathbf{I}_n \otimes [\mathbf{v}]_\times)\,\mathbf{p} \\ \mathbf{1}_n \end{bmatrix}\right\}, & \text{if } d = 3 \end{cases} \tag{19}$$

and, since $\dim(\mathcal{R}_{\circlearrowleft}) = 1$ for $d \in \{2, 3\}$, the trivial variations set has dimension $q_t = d + 2$. This formally results

$$\mathcal{S}_t = \mathrm{span}\left\{\begin{bmatrix} \mathbf{1}_n \otimes \mathbf{I}_d \\ \mathbf{0}_n \end{bmatrix}, \begin{bmatrix} \mathbf{p} \\ \mathbf{0}_n \end{bmatrix}, \mathcal{R}_{\circlearrowleft}\right\}. \tag{20}$$

Adopting the unified view of Section IV, it is convenient to distinguish between the differential manifolds $\mathbb{R}^2 \times \mathbb{S}^1 = SE(2)$ and $\mathbb{R}^3 \times \mathbb{S}^1$. Indeed, in the first case ($d = 2$), assuming that the formation evolves on the (xy)-plane of $\mathscr{F}_W$ identified by $\mathbf{e}_1 \times \mathbf{e}_2$, we have that $\mathbf{U}_{ij} = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{0}_3\end{bmatrix}$ for all the measurements, and that $\mathbf{R}_i = \mathbf{R}(\theta_i, \mathbf{e}_3)\ \forall i \in \{1 \dots n\}$, according to the axis-angle representation $\mathbf{R}(\cdot,\cdot)$ of a 3D rotation, hence, $\mathbf{V}_{ij} = \begin{bmatrix}\mathbf{0}_{3\times 2} & \mathbf{e}_3\end{bmatrix}$ in correspondence to any edges of $\mathcal{G}$. In the second case ($d = 3$), instead, $\mathbf{U}_{ij} = \mathbf{I}_3$, while $\mathbf{R}_i = \mathbf{R}(\theta_i, \mathbf{v})$, and, thus, $\mathbf{V}_{ij} = \begin{bmatrix}\mathbf{0}_{3\times 2} & \mathbf{v}\end{bmatrix}$. In particular, one can check that $\begin{bmatrix}\mathbf{0}_{3\times 2} & [(\bar{\mathbf{p}}_{ij}^\perp)^\top \ 0]^\top\end{bmatrix} = [[\bar{\mathbf{p}}_{ij}^\top \ 0]^\top]_\times \begin{bmatrix}\mathbf{0}_{3\times 2} & \mathbf{e}_3\end{bmatrix}$ and $\begin{bmatrix}\mathbf{0}_{3\times 2} & [\bar{\mathbf{p}}_{ij}]_\times \mathbf{v}\end{bmatrix} = [\bar{\mathbf{p}}_{ij}]_\times \begin{bmatrix}\mathbf{0}_{3\times 2} & \mathbf{v}\end{bmatrix}$. Thus, we conclude that the expression (10) can be reduced to the bearing rigidity matrix (17). Furthermore, when $\bar{\mathcal{D}} = (\mathbb{R}^d \times \mathbb{S}^1)^n$, the set $\mathcal{S}_v$ includes the $2n$ unfeasible rotational movements of the agents, in addition to the translation along the z-axis of $\mathscr{F}_W$ when $d = 2$. Summing up, concerning the virtual trivial variations, we have that $q_v = 3n$ when $d = 2$ and $q_v = 2n$ when $d = 3$.

3) Rigidity Theory in $SE(3)$: The recent work [31] accounts for formations of agents equipped with a bearing sensor whose configuration is given by an element of $SE(3)$. An example is given by a swarm of fully actuated aerial platforms provided with on-board omnidirectional cameras. The considered group of agents can be modeled as a framework $(\mathcal{G}, \chi)$ in $\bar{\mathcal{D}} = SE(3)^n$, where $\mathcal{G}$ is a *directed* graph and χ deals with the *positions vector* $\mathbf{p} = \begin{bmatrix}\mathbf{p}_1^\top \dots \mathbf{p}_n^\top\end{bmatrix}^\top \in \mathbb{R}^{3n}$, and the $(3n \times 3)$ *attitudes matrix* $\mathbf{R}_a = \begin{bmatrix}\mathbf{R}_1^\top \dots \mathbf{R}_n^\top\end{bmatrix}^\top \in SO(3)^n$, stacking all the agent position vectors and rotation matrices, respectively. Note that the commands space $\mathcal{I}$ of each ith agent includes its linear velocity $\dot{\mathbf{p}}_i \in \mathbb{R}^3$ and its angular velocity $\boldsymbol{\omega}_i \in \mathbb{R}^3$, both expressed in $\mathscr{F}_W$. Given these premises, it is thus possible to prove that the bearing rigidity matrix, belonging to $\mathbb{R}^{3m\times 6n}$, turns out to be

$$\mathbf{B}_{\mathcal{G}}(\chi) = \begin{bmatrix}\mathrm{diag}(s_{ij}\mathbf{R}_i^\top \mathbf{P}(\bar{\mathbf{p}}_{ij}))\bar{\mathbf{E}}^\top & -\mathrm{diag}(\mathbf{R}_i^\top [\bar{\mathbf{p}}_{ij}]_\times)\bar{\mathbf{E}}_o^\top\end{bmatrix}. \tag{21}$$

Comparing (21) with (17), we observe that the translation, uniform scaling, and coordinated rotation are trivial variations also for a framework $(\mathcal{G}, \chi)$ in $SE(3)^n$, however, the concept of coordinated rotation has to be redefined since the agents orientation is no longer controllable only via a single angle.

Specifically, it has been proven in [31]

$$\mathcal{R}_{\circlearrowleft} = \mathrm{span}\left\{\left\{\begin{bmatrix} (\mathbf{I}_n \otimes [\mathbf{e}_h]_\times)\,\mathbf{p} \\ \mathbf{1}_n \otimes \mathbf{e}_h \end{bmatrix}\right\}_{h=1,2,3}\right\}. \tag{22}$$

Therefore, the following set $\mathcal{S}_t$ tha $\dim(\mathcal{S}_t) = 7$:

$$\mathcal{S}_t = \mathrm{span}\left\{\begin{bmatrix} \mathbf{1}_n \otimes \mathbf{I}_3 \\ \mathbf{0}_{3n} \end{bmatrix}, \begin{bmatrix} \mathbf{p} \\ \mathbf{0}_{3n} \end{bmatrix}, \mathcal{R}_{\circlearrowleft}\right\}. \tag{23}$$

When $\bar{\mathcal{D}} = SE(3)^n$, each agent is characterized by $c = 6$ DOFs and, thus, it can vary its position and attitude in any direction of the 3D space. Therefore, it follows that $\mathbf{U}_{ij} = \mathbf{V}_{ij} = \mathbf{I}_3$ and the matrix $\mathbf{B}_{\mathcal{G}}^+(\chi)$ corresponds to that in (21). Clearly, when the agents act in $SE(3)$, no embedding in an higher dimensional manifold is needed, which translates into the absence of virtual trivial motions, and, correspondingly, no null columns appear in $\mathbf{B}_{\mathcal{G}}^+(\chi)$.

B. Rigidity Properties

Accounting for Theorem 2, the next result provides a necessary and sufficient condition to check if a given homogeneous formation is IBR. Its validity is guaranteed by observing that $\mathrm{rk}(\mathbf{B}_{\mathcal{G}}(\chi(t))) = \mathrm{rk}(\mathbf{B}_{\mathcal{G}}^+(\chi(t)))$ together with the results of [26, Th. 4], [28, Th. III.6], and [31, Th. 3] dealing with homogeneous frameworks embedded in $\mathbb{R}^{dn}$, $(\mathbb{R}^d \times \mathbb{S}^1)^n$, $SE(3)^n$, respectively.

Theorem 3: A homogeneous framework $(\mathcal{G}, \chi(t))$ in $\bar{\mathcal{D}}$, where each agent has $c \leq 6$ DOFs is IBR if and only if $\mathrm{rk}(\mathbf{B}_{\mathcal{G}}^+(\chi(t))) = cn - c - 1$.

Proof: Note that $\mathrm{rk}(\mathbf{B}_{\mathcal{K}}^+(\chi(t))) = 6n - q_t - q_v$, where the difference $6n - q_v$ corresponds to the total DOFs of the formations, namely, to $c_{\text{tot}} = cn \leq 6n$ in homogeneous cases. On

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the other hand, from (16), (20), and (23), it can be inferred that for a homogeneous framework it holds that $q_t = c + 1$, i.e., shape preservation is ensured when the formation acts as a unique rigid body having c DOFs and when it scales. Theorem 2, thus, ensures that a noncolinear homogeneous formation is IBR if and only if $\mathrm{rk}(\mathbf{B}^+_{\mathcal{G}}(\chi(t))) = \mathrm{rk}(\mathbf{B}^+_{\mathcal{K}}(\chi(t))) = (6n - q_v) - q_t = cn - (c+1)$. ■

As already highlighted in Section III-A, the IBR property is generally studied for dynamic frameworks, while the BR and the GBR ones are usually discussed for static frameworks. Nonetheless, these last two properties can be also stated for dynamic frameworks over time, and in particular they can hold for any t.

Given these premises, the next theorem provides a complete characterization to clarify the relation between BR, GBR, and IBR properties for noncolinear frameworks embedded in any differential manifold $\bar{\mathcal{D}}$.

Theorem 4: Given a differential manifold $\bar{\mathcal{D}}$ for any t

i) a framework $(\mathcal{G}, \chi(t))$ is IBR if and only if it is BR;

ii) a framework $(\mathcal{G}, \chi(t))$ is IBR if it is GBR.

Proof: $i)$ Proceeding by contrapositive, if $(\mathcal{G}, \chi(t))$ is not BR for any t, there exists at least a configuration $\chi(t')$ in the neighborhood $\mathcal{U}(\chi)$ of $\chi(t)$, such that $\chi(t') \in \mathcal{Q}(\chi(t)) \setminus \mathcal{C}(\chi(t))$, and therefore the framework $(\mathcal{G}, \chi(t))$ would result to be IBF according to the consequence of Definition 12. To prove the reverse, we assume that the framework $(\mathcal{G}, \chi(t))$ is BR for any t. *Ab absurdo*, if the framework was not IBR (IBF) there would be a variation deforming $\chi(t)$ into $\chi(t') \in \mathcal{Q}(\chi(t)) \setminus \mathcal{C}(\chi(t))$, implying a contradiction with respect to Definition 7.

$ii)$ It is a direct implication of Proposition 1 and i). ■

The following results can also be stated for the frameworks embedded in $\bar{\mathcal{D}} = \mathbb{R}^{dn}$, $d \in \{2, 3\}$.

Proposition 3: A BR framework $(\mathcal{G}, \chi(t))$ in $\bar{\mathcal{D}} = \mathbb{R}^{dn}$, $d \in \{2, 3\}$, is also GBR.

Proof: According to [26, Th. 1 and 3], a framework $(\mathcal{G}, \chi(t'))$ is BE/BC to $(\mathcal{G}, \chi(t))$ if and only if its corresponding positions vector belongs to $\ker(\mathbf{B}_{\mathcal{G}}(\chi(t)))/\ker(\mathbf{B}_{\mathcal{K}}(\chi(t)))$, moreover, for a BR framework it holds that $\ker(\mathbf{B}_{\mathcal{G}}(\chi(t))) \subseteq \ker(\mathbf{B}_{\mathcal{K}}(\chi(t)))$. These two facts imply that a framework $(\mathcal{G}, \chi(t))$ in $\bar{\mathcal{D}} = \mathbb{R}^{dn}$, $d \in \{2, 3\}$, is GBR if the condition $\ker(\mathbf{B}_{\mathcal{G}}(\chi(t))) = \ker(\mathbf{B}_{\mathcal{K}}(\chi(t)))$ holds. Hence, because of Theorem 1, the bearing rigidity property ensures the global rigidity property. ■

We observe that the requirement on the null spaces equivalence for GBR property derived in the proof of Proposition 3 coincides with the definition of IBR property provided in Definition 12. Thus, a IBR framework in $\mathbb{R}^{dn}$ is also GBR for any time and vice versa, leading to the next corollary.

Corollary 1: For frameworks embedded in $\bar{\mathcal{D}} = \mathbb{R}^{dn}$, $d \in \{2, 3\}$, bearing rigidity, global bearing rigidity, and infinitesimal bearing rigidity are equivalent properties.

The equivalence among rigidity properties is not valid when introducing the RDOFs as for frameworks in $(\mathbb{R}^d \times \mathbb{S}^1)^n$, $d \in \{2, 3\}$, or in $SE(3)^n$. Fig. 2 provides an example [panel (a)] of a IBR framework in $(\mathbb{R}^2 \times \mathbb{S}^1)^n$ composed of $n = 4$ agents, that is not GBR since it is possible to deform into a BE and not BC framework [panel (b)].

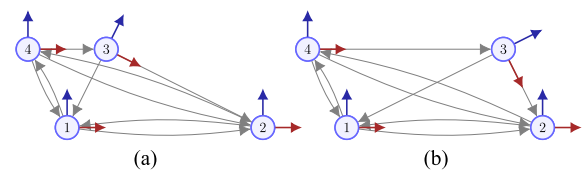

Fig. 2. Example of IBR and not GBR framework in $\bar{\mathcal{D}} = (\mathbb{R}^2 \times \mathbb{S}^1)^n$ with $n = 4$. Blue circles, blue-red arrows, and gray arrows denote, respectively, the agents, the corresponding local frames, and the existing bearing measurements.

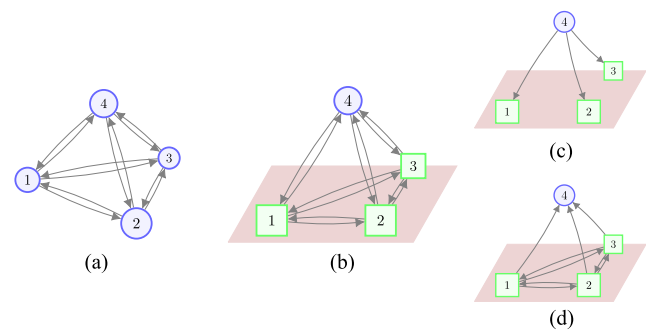

Fig. 3. Sensing graphs for the considered case studies. (a) Homogeneous formation. (b) Heterogeneous formation $\mathcal{K} = \mathcal{G}_1 \cup \mathcal{G}_2$, with (c) aerial bearing subgraph $\mathcal{G}_1$, (d) terrestrial bearing subgraph $\mathcal{G}_2$.

VI. Applications of Unified Rigidity Theory

In these case studies, time dependency is omitted for brevity.

A. Homogeneous Formation in $\mathbb{R}^3 \times \mathbb{S}^2$ Case Study

As a first application of unified rigidity theory, we discuss a homogeneous formation case study that has not been considered in the literature and show how this can be well accommodated in the framework of Section IV.

We account for the case study in Fig. 3(a), where a formation is composed of four aerial platforms (blue circle) with bearings coming from gimbal cameras, which, despite the free movement of the agents in the 3D space, keep their image planes aligned with the horizon and are denied the roll motion: For each agent $c = 5$ ($c^t = 3$ and $c^r = 2$), and $\mathcal{D} = \mathbb{R}^3 \times \mathbb{S}^2$. The considered $n = 4$ agents group is modeled as a homogeneous agent framework $(\mathcal{K}, \chi)$, where $\chi = (\chi_1 \dots \chi_4)$ with $\chi_1, \chi_2, \chi_3, \chi_4 \in \mathbb{R}^3 \times \mathbb{S}^2$. Note that the relative bearing measurement between each pair of agents is expressed in their local frames and the underlying graph is complete, thus the formation is IBR, according to Definition 12.

According to Proposition 2 and the related discussion held in Section V for the homogeneous case, all the edges can vary in the 3D space in the same way, in particular their rotations are constrained around the y-axis and z-axis of $\mathscr{F}_W$. Hence, $\mathbf{U}_{ij} = \mathbf{I}_3$ and $\mathbf{V}_{ij} = [\mathbf{0}_3\ \mathbf{e}_2\ \mathbf{e}_3]$ in correspondence to any $e_{ij} \in \mathcal{E}$. The bearing matrix $\mathbf{B}^+_{\mathcal{K}}(\chi) \in \mathbb{R}^{36 \times 24}$ takes the form as in Fig. 4(a), where four null columns can be highlighted.

From Theorem 3, it holds that $\mathrm{rk}(\mathbf{B}^+_{\mathcal{K}}(\chi)) = 4(3+2) - (3+2) - 1 = 14$. This fact can be numerically verified and implies that $\mathrm{null}(\mathbf{B}^+_{\mathcal{K}}(\chi)) = q_t + q_v = 10$. In particular, we can identify

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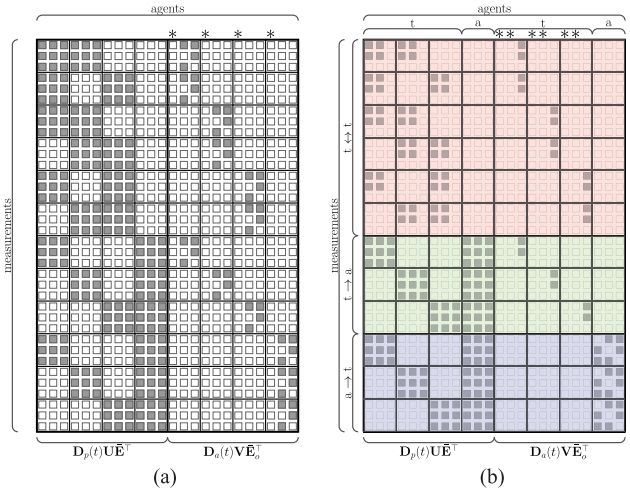

Fig. 4. Structure of the bearing rigidity matrix for the considered case studies. (a) Homogeneous and (b) heterogeneous formations. Dark squares indicate nonnull values of the matrix; in the heterogeneous case t and a labels refer, respectively, to terrestrial and aerial agents. Null columns are indicated with an asterisk.

TABLE III
MATRICES FOR THE HETEROGENEOUS FORMATION CASE STUDY

(i,j)	$\mathbf{U}_{ij}$	$\mathbf{V}_{ij}$
$(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)$	$[\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{0}_3]$	$[\mathbf{0}_{3\times 2}\ \mathbf{e}_3]$
$(1,4),(2,4),(3,4)$	$\mathbf{I}_3$	$[\mathbf{0}_{3\times 2}\ \mathbf{e}_3]$
$(4,1),(4,2),(4,3)$	$\mathbf{I}_3$	$\mathbf{I}_3$

$q_t = c + 1 = 6$ trivial variations, which are three translations, two coordinated rotations, and one scaling, in addition to $q_v = 4$ virtual variations that are due to the four null columns of $\mathbf{B}^+_{\mathcal{K}}(\chi)$ and correspond to the actual constraints on the agents motion.

B. Heterogeneous Formation Case Study

The discussion carried out in Section IV about the structure of the rigidity matrix turns out to be particularly useful for the case of heterogeneous formations made up of agents whose configurations evolve in different differential manifolds.

To validate this observation, we account for the case study in the right part of Fig. 3 [panels (b)–(d)] referring to a formation involving three unicycle-modeled terrestrial robots (green squares), and a fully actuated aerial platform (blue circle). We assume that the agents are able to retrieve relative bearings expressed in their local frames according to the depicted complete graph $\mathcal{K}$. Hence, the considered $n = 4$ agents group is modeled as the IBR framework $(\mathcal{K}, \chi)$, where $\chi = (\chi_1 \dots \chi_4)$ with $\chi_1, \chi_2, \chi_3 \in \mathbb{R}^2 \times \mathbb{S}^1$, and $\chi_4 \in SE(3)$.

Assuming that the global frame is oriented so that the terrestrial vehicles can move on the (xy)-plane of $\mathscr{F}_W$, the matrices $\mathbf{U}$ and $\mathbf{V}$ in (10) are determined based on Table III.

Hence, the rigidity matrix $\mathbf{B}^+_{\mathcal{K}}(\chi) \in \mathbb{R}^{36\times 24}$ has the structure reported in Fig. 4(b).

The set $\mathcal{S}_t$ counts $q_t = 5$ trivial variations consisting of the translation of the whole formation on the (xy)-plane of the global frame, its coordinated rotation around the z-axis of $\mathscr{F}_W$, and its scaling (indeed, the third, sixth, ninth, tenth, and eleventh columns of $\mathbf{B}^+_{\mathcal{K}}(\chi)$ results to be linearly dependent with respect to the remaining ones). In addition, the matrix has six null columns, meaning that the set $\mathcal{S}_v$ of virtual variations has cardinality $q_v = 6$. This, in fact, coincides with the (unfeasible) rotations of the terrestrial robots around their x and y-axis, and is related to the zero columns in $\mathbf{B}^+_{\mathcal{K}}(\chi)$. From these premises, it follows that $\mathrm{null}(\mathbf{B}^+_{\mathcal{K}}(\chi)) = 11$ and, finally, that $\mathrm{rk}(\mathbf{B}^+_{\mathcal{K}}(\chi)) = 13$, as can be numerically verified.

To conclude, we note that the given IBR formation results from the union of two sensing graphs, each one corresponding with the measurements obtained by either the terrestrial robots $(\mathcal{G}_1)$ or the aerial platform $(\mathcal{G}_2)$, as in Fig. 3(c) and (d). Focusing on the infinitesimal rigidity of the resulting two frameworks, we can observe that any rotation of the aerial vehicle is an infinitesimal variation for $(\mathcal{G}_1, \chi)$, and similarly, any rotation of the terrestrial vehicle is an infinitesimal variation for $(\mathcal{G}_2, \chi)$, concluding that the two frameworks are not IBR differently from their union $(\mathcal{K}, \chi)$.

VII. On Colinear Formations

In this section we briefly discuss the colinear formations case, focusing on the bearing-preserving variations set. According to Definition 2, a formation composed of $n \geq 3$ distinctly placed agents is *colinear* if all the agents are collinear, i.e., for any kth component of the position vectors, $k \in \{1 \dots d\}$, it exists $c \in \mathbb{R}$, such that $p_i^k = c p_j^k$ for each pair (v_i, v_j) of agents in the group. Under this hypothesis, we can observe that the shape uniqueness is guaranteed for a larger set of infinitesimal variations with respect to that described in the previous sections. Although this statement is valid independently on $\bar{\mathcal{D}}$, in the following we distinguish between the three cases previously treated.

$\bar{\mathcal{D}} = \mathbb{R}^{dn}$, $d \in \{2, 3\}$. For a formation composed of n agents, controllable in $\mathbb{R}^d$, $d \in \{2, 3\}$, and aligned along a certain direction identified by the unit vector $\mathbf{w} \in \mathbb{S}^{d-1}$ the bearing measurements are collinear, namely, $\mathbf{b}_{\mathcal{G}}(\chi) = \mathrm{diag}(b_1\mathbf{I}_d \cdots b_m\mathbf{I}_d)(\mathbf{1}_m \otimes \mathbf{w})$, with $b_i \in \mathbb{R}$ for $i \in \{1 \dots m\}$. Thus, these are preserved despite the displacement of any agent along the direction specified by $\mathbf{w}$ and the translation of the whole formation in the subspace $\mathcal{W}$ of $\mathbb{R}^d$ orthogonal to $\mathbf{w}$. Hence the trivial variation set[4] coincides with $\mathcal{S}_t^d = \mathrm{span}\{\mathbf{I}_n \otimes \mathbf{w}, \mathbf{1}_n \otimes \mathbf{W}\}$, where $\mathbf{W} \in \mathbb{R}^{d\times(d-1)}$ is a matrix whose columns represent a basis for $\mathcal{W}$. Trivially, $\mathcal{S}_t^d$ has dimension $n + (d - 1) > d + 1 = |\mathcal{S}_t|$, with $\mathcal{S}_t$ as in (16).

$\bar{\mathcal{D}} = (\mathbb{R}^d \times \mathbb{S}^1)^n$, $d \in \{2, 3\}$. For an n-agent formation acting in $(\mathbb{R}^d \times \mathbb{S}^1)^n$, the bearing measurements are retrieved in the local agents frame. Additionally, each agent has a (controllable) RDOF allowing rotations only around the direction of $\mathbf{v} \in \mathbb{S}^2$ when $d = 3$. To analyze the colinear situation in which all the agents are aligned along the direction identified by the unit vector $\mathbf{w} \in \mathbb{S}^{d-1}$, it is necessary to distinguish between the following cases: $i)$ $d = 2$, or $d = 3$ and $\mathbf{v} \neq \mathbf{w}$, $ii)$ $d = 3$ and $\mathbf{v} = \mathbf{w}$. For a colinear formation satisfying conditions $i)$, the bearing measurements are preserved when the whole agents group translates along any direction in the $(d-1)$-D subspace $\mathcal{W} \subseteq \mathbb{R}^d$

[4]Note that the uniform scaling of the formation corresponds to suitable (not equal) translations of all the agents along the direction $\mathbf{v} \in \mathbb{S}^{d-1}$.

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orthogonal to $\mathbf{w}$, when a coordinated rotation is performed according to the definition given in Section V-A2, and also when any agent moves along the alignment direction. Therefore, the trivial variation set $\mathcal{S}_t^d$ is spanned by $n+(d-1)+1$ elements. In case $ii)$, the dimension of $\mathcal{S}_t^d$ increases since the formation is not required to perform a coordinated rotation to preserve the bearings; also the rotation of any agent around the axis identified by $\mathbf{v}=\mathbf{w}$ ensures the measurements maintenance. Hence, we get $|\mathcal{S}_t^d|=2n+(d-1)$. Note that in both cases $i)$ and $ii)$ the trivial variation set has dimension greater with respect to noncolinear case for which $|\mathcal{S}_t|=d+2$ according to (20).

$\bar{\mathcal{D}}=SE(3)^n$. When the space of interest is $\bar{\mathcal{D}}=SE(3)^n$, we figure out that for the colinear case in which the n agents are aligned along the direction identified by the unit vector $\mathbf{w}\in\mathbb{S}^2$, bearings are preserved when any agent translates or rotates along the direction of $\mathbf{v}$ and when the whole formation performs a translation or a coordinated rotation around any direction in the (two-dimensional) subspace $\mathcal{W}\subseteq\mathbb{R}^3$ orthogonal to $\mathbf{w}$. The trivial variation set has thus dimension $2n+4>7=|\mathcal{S}_t|$ with $\mathcal{S}_t$ as in (20).

In general, we can observe that for every differential manifold $\bar{\mathcal{D}}$ it occurs that $|\mathcal{S}_t^d|>|\mathcal{S}_t|$.

VIII. Conclusion

This work focuses on bearing rigidity theory applied to multiagent systems whose elements are characterized by a certain number of both TDOFs and RDOFs. As original contribution, we propose a general framework for the definition of the main rigidity properties without accounting for the specific controllable agents state domain. Moreover, we summarize the existing results about bearing rigidity theory for frameworks embedded in $\mathbb{R}^d$, in $\mathbb{R}^d\times\mathbb{S}^1$ with $d\in\{2,3\}$, and in $SE(3)$. For each case, the principal definitions are provided and the infinitesimal rigidity property is investigated by deriving a necessary and sufficient condition based on the rigidity matrix rank. In addition, we provide a necessary and sufficient condition to check IBR property of a given system independently on its differential manifold. This arises from the derivation of a unified structure of the rigidity matrix that does not rest on the specific agents domain but exploits the fact that $\mathcal{D}\subseteq SE(3)$.

Appendix

Theorem 5: For a noncolinear framework $(\mathcal{K},\chi)$ in $\mathbb{R}^{dn}$, $d\in\{2,3\}$, it holds that $\ker(\mathbf{B}_{\mathcal{K}}(\chi))=\mathcal{S}_t$, or equivalently, $\mathrm{rk}(\mathbf{B}_{\mathcal{K}}(\chi))=dn-d-1$, where $\mathcal{S}_t$ corresponds to the trivial variation set (16).

Proof: The bearing rigidity matrix associated to $(\mathcal{K},\chi)$ is so that the kth row block, corresponding to $e_k=(v_i,v_j)\in\mathcal{E}_{\mathcal{K}}$ with $i<j$,[5] has the following form, where $\mathbf{0}_{p\times q}$ is the $(p\times q)$ zero matrix and $\mathbf{B}_{ij}=s_{ij}\mathbf{P}(\bar{\mathbf{p}}_{ij})\in\mathbb{R}^{d\times d}$

$$\begin{bmatrix}\mathbf{0}_{d\times d(i-1)} & -\mathbf{B}_{ij} & \mathbf{0}_{d\times d(j-i-1)} & \mathbf{B}_{ij} & \mathbf{0}_{d\times d(n-j)}\end{bmatrix}. \quad (24)$$

[5]For undirected graph, it is always possible to choose a suitable edges labeling ensuring the desired requirement.

For $d=2$, $\mathbf{B}_{ij}=s_{ij}^3\mathbf{r}_{ij}\mathbf{r}_{ij}^\top$, with $\mathbf{r}_{ij}=[p_{ij}^y\ \ -p_{ij}^x]^\top\in\mathbb{R}^2$, where $p_{ij}^x,p_{ij}^y\in\mathbb{R}$ are the (scalar) components of vector $\mathbf{p}_{ij}\in\mathbb{R}^2$ along the x-axis and y-axis of the global frame, respectively. Note that $\mathbf{B}_{ij}$ is neither zero nor full-rank, hence the k-row block (24) has unitary rank. For this reason, for each edge $e_k=(v_i,v_j)\in\mathcal{E}_{\mathcal{K}}$ with $i<j$, we consider the next opportunely scaled version of (24)

$$\begin{bmatrix}\mathbf{0}_{1\times 2(i-1)} & -\mathbf{r}_{ij}^\top & \mathbf{0}_{1\times 2(j-i-1)} & \mathbf{r}_{ij}^\top & \mathbf{0}_{1\times 2(n-j)}\end{bmatrix} \quad (25)$$

obtaining the matrix $\mathbf{B}(n)\in\mathbb{R}^{((n-1)n/2)\times 2n}$. This has the same rank of $\mathbf{B}_{\mathcal{K}}(\chi)$ but lower dimensions, so hereafter, we consider $\mathbf{B}(n)$ instead of $\mathbf{B}_{\mathcal{K}}(\chi)$ and we prove thesis by induction on the number n of agents in the formation.

Base case: $n=3$

We aim at proving that $\mathrm{rk}(\mathbf{B}(3))=3$. To do so, observe that

$$\mathbf{B}(3)=\begin{bmatrix}-\mathbf{r}_{12}^\top & \mathbf{r}_{12}^\top & \mathbf{0}_{1\times 2}\\ -\mathbf{r}_{13}^\top & \mathbf{0}_{1\times 2} & \mathbf{r}_{13}^\top\\ \mathbf{0}_{1\times 2} & -\mathbf{r}_{23}^\top & \mathbf{r}_{23}^\top\end{bmatrix}\in\mathbb{R}^{3\times 6} \quad (26)$$

is full-rank whether the agents are not all collinear. Because of noncolinear formation hypothesis the thesis is proved.

Inductive step $n=\bar{n}$: Note that, given a set of $\bar{n}$ agents, for each subset containing $\bar{n}-1$ elements, it is possible to partition $\mathbf{B}(\bar{n})$ so that

$$\mathbf{B}(\bar{n})=\begin{bmatrix}\mathbf{B}(\bar{n}-1) & \mathbf{1}_\eta\otimes\mathbf{0}_2^\top\\ \mathrm{diag}\{\mathbf{r}_{i\bar{n}}^\top\}_{i=1}^{(\bar{n}-1)} & \mathrm{col}\{\mathbf{r}_{i\bar{n}}^\top\}_{i=1}^{(\bar{n}-1)}\end{bmatrix}\begin{matrix}\text{1st block}\\ \text{2nd block}\end{matrix} \quad (27)$$

where the first block has $\eta=(\bar{n}-1)(\bar{n}-2)/2$ rows related to the edges incident to the first $\bar{n}-1$ agents, while the second block has $\bar{n}$ rows related to the edges connecting the $\bar{n}$th agent with the first $\bar{n}-1$ agents. For inductive hypothesis the thesis holds for $\bar{n}-1\geq 3$, i.e., $\mathrm{rk}(\mathbf{B}(\bar{n}-1))=2\bar{n}-5$, thus, the first block of $\mathbf{B}(\bar{n})$ in (27) contains $2\bar{n}-5$ linearly independent rows. Moreover, there are at least two agents, for instance the ith and jth agent, that are not aligned with the $\bar{n}$th agent, hence it does not exist $c\in\mathbb{R}$, such that $\mathbf{r}_{i\bar{n}}=c\mathbf{r}_{j\bar{n}}$ and the rows related to the edges $(v_i,v_{\bar{n}})$, and $(v_j,v_{\bar{n}})$ are linearly independent with respect to the rows of the first block. $\mathbf{B}(\bar{n})$ has thus at least $2\bar{n}-3$ linearly independent rows, and, since $\mathrm{rk}(\mathbf{B}(\bar{n}))\leq 2\bar{n}-3$ for Lemma 4 in [26], then it must be $\mathrm{rk}(\mathbf{B}(\bar{n}))=2\bar{n}-3$ concluding the proof for the case $d=2$.

When $d=3$, the matrix $\mathbf{B}_{ij}$ in (24) turns out to be

$$\mathbf{B}_{ij}=s_{ij}^3\begin{bmatrix}(p_{ij}^y)^2+(p_{ij}^z)^2 & -p_{ij}^xp_{ij}^y & -p_{ij}^xp_{ij}^z\\ -p_{ij}^yp_{ij}^x & (p_{ij}^x)^2+(p_{ij}^z)^2 & -p_{ij}^yp_{ij}^z\\ -p_{ij}^zp_{ij}^x & -p_{ij}^zp_{ij}^y & (p_{ij}^x)^2+(p_{ij}^y)^2\end{bmatrix} \quad (28)$$

where $p_{ij}^x,p_{ij}^y,p_{ij}^z\in\mathbb{R}$ are the (scalar) components of vector $\mathbf{p}_{ij}\in\mathbb{R}^3$ along the x-axis, y-axis, and z-axis of the $\mathscr{F}_W$, respectively. The proof for this case thus follows the same inductive reasoning performed for $d=2$. ■

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Giulia Michieletto (Member, IEEE) received the M.S. degree in Automation Engineering and the Ph.D. degree in Information Engineering from the University of Padova, Padova, Italy, in 2014 and 2018.

She is currently an Assistant Professor with the Department of Management and Engineering, University of Padova. From March 2016 to February 2017, she was a Visiting Researcher with LAAS-CNRS, Toulouse, France. From February 2018 to November 2019, she was a Postdoc Fellow with the SPARCS group, University of Padova, Italy. Her research interests include multiagent systems modeling and control with a special regard to networked formations of aerial vehicles and nano satellites.

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Angelo Cenedese (Member, IEEE) received the M.S. degree in Electronic Engineering and the Ph.D. degree in Energy Science from the University of Padova, Italy, in 1999 and 2004.

He is currently an Associate Professor with the Department of Information Engineering, University of Padova, and Leader of the SPARCS research group. He has held several visiting positions with the UKAEA-JET laboratories, U.K., the UCLA Vision Lab, CA, USA, the F4E European Agency, Spain. He has authored or coauthored more than 170 papers and holds three patents in the areas of his interests. His research interests include system modeling, control theory and its applications, sensor and actuator networks, multi agent systems.

Daniel Zelazo received the BSc. and M.Eng degrees in electrical engineering from the Massachusetts Institute of Technology, Cambridge, MA, USA, in 1999 and 2001, respectively, the Ph.D. degree in aeronautics and astronautics, from the University of Washington, Seattle, WA, USA, in 2009.

He is an Associate Professor of Aerospace Engineering with the Technion - Israel Institute of Technology, Haifa, Israel. From 2010–2012, he served as a Postdoctoral Research Associate and Lecturer with the Institute for Systems Theory & Automatic Control in the University of Stuttgart, Stuttgart, Germany. His research interests include topics related to multiagent systems, optimization, and graph theory.

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