A Network Optimization Approach to Cooperative Control Synthesis

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***Abstract*—The mathematical theory of nonlinear cooperative control relies heavily on notions from graph theory and passivity theory. A general analysis result is known about cooperative control of maximally equilibrium-independent systems, relating steady-states of the closed-loop system to network optimization theory. However, until now only analysis results have been proven, and there is no known synthesis result. This letter presents a controller synthesis procedure for a class of diffusively coupled dynamic networks. We use tools from network optimization and convex analysis to show that for a network composed of maximally equilibrium independent passive systems, it is possible to construct controllers on the edges that are maximally equilibrium independent output-strictly passive and achieve any desired formation. Furthermore, we show that this can be achieved with linear controllers. We also provide a simple controller augmentation procedure to allow for reconfiguration of the desired output formation without a redesign of the nominal control. We then apply the presented methods to reconstruct the well-known consensus algorithm, and to study formation control in networks of damped oscillators.**

***Index Terms*—Multi-agent systems, cooperative systems, nonlinear control systems, optimization**

I. Introduction

THE STUDY of multi-agent networks has been in the pinnacle of control research for the last few years, exhibiting both a rich theoretical framework as well as a wide range of applications [1]–[3]. An important problem in the study of multi-agent system is that of controller synthesis - namely the construction of distributed controllers that ensures the closed-loop system converges to some desired output. This control goal encompasses many canonical problems including synchronization and formation control [4]–[7].

In this venue, researchers have tried to establish a unified theory for networks of dynamical systems, and in many of them passivity theory plays a major role [8]. Passivity theory allows for the analysis of dynamic networks to be separated into complimentary layers - the *dynamic system* layer and the *information exchange network* layer. The use of passivity theory to study the convergence properties of these networked systems was originally proposed in [9]. Many variations and extensions of this theme have been explored in a variety of contexts. For example, the related concepts of incremental passivity or relaxed co-coercivity have been used to study various synchronization problems [10], [11], and more general frameworks including Port-Hamiltonian systems on graphs [12]. Passivity is also widely used in coordinated control of robotic systems [13] and the teleoperation of UAV swarms [14].

In the recent work [15], the passivity approach to cooperative control was revealed to have another interpretation related to a class of network optimization problems. Network optimization, a subdomain of convex optimization theory, deals with optimization of functions defined over graphs [16]. The main result of [15] showed that the asymptotic behavior of these networked systems is *(inverse) optimal* with respect to a family of network optimization problems. In fact, the steady-state input-output signals of both the dynamical systems and the controllers comprising the networked system can be associated to the optimization variables of either an *optimal flow* or an *optimal potential* problem; these are the two canonical dual network optimization problems described in [16].

The results of [15], however, are an *analysis* result. For such analysis to be practically useful, one must also develop synthesis procedures to design controllers for networked systems to achieve the desired coordination goals. This motivates the main objective of this letter. The goal of this letter is to describe a solution to the synthesis problem for multi-agent systems by applying results from [15]. Our control objective is to assure the convergence of the networked system to a desired relative output configuration - we term this desired output a *formation* of the system.[1] Our main contributions are as follows:

i) We prove that under certain conditions on the networked system, any desired formation can be achieved as a steady-state of the system.
ii) Given a collection of desired relative outputs (the formations), we present a synthesis procedure for controllers ensuring the closed-loop system globally asymptotically converges to the desired formations.

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[1]This is not to be confused with the standard formation control problem which aims to control a team of agents to some desired spatial configuration [5]. Our use of the term formation in this context is more abstract.

iii) We propose a simple controller augmentation procedure to allow for reconfiguration of the desired output formation without a redesign of the nominal control. This augmentation is realized by designing certain constant exogenous inputs for the controller without modifying its dynamic structure.

The rest of the letter is as follows. Section II reviews the necessary results from [15]. Section III derives the synthesis methods for the controllers, as well as the "formation reconfiguration" scheme. Lastly, Section IV studies the case of simple integrator agents, reconstructing the well-known consensus protocol [4], and also demonstrates the formation reconfiguration scheme.

Notations: This letter employs basic notions from algebraic graph theory [17]. An undirected graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ consists of a finite set of vertices $\mathbb{V}$ and edges $\mathbb{E} \subset \mathbb{V} \times \mathbb{V}$. We denote by $k = \{i, j\} \in \mathbb{E}$ the edge that has ends i and j in $\mathbb{V}$. For each edge k, we pick an arbitrary orientation and denote $k = (i, j)$ when $i \in \mathbb{V}$ is the *head* of edge k and $j \in \mathbb{V}$ the *tail*. The incidence matrix of $\mathcal{G}$, denoted $\mathcal{E} \in \mathbb{R}^{|\mathbb{E}|\times|\mathbb{V}|}$, is defined such that for edge $k = (i, j) \in \mathbb{E}$, $[\mathcal{E}]_{ik} = +1$, $[\mathcal{E}]_{jk} = -1$, and $[\mathcal{E}]_{\ell k} = 0$ for $\ell \neq i, j$.

Furthermore, we also use a few basic notations from linear algebra. For a linear map $T: U \to V$ between vector spaces, we denote the kernel of T by $\ker T$, and the image of T by $\mathrm{Im}(T)$. For a subspace U of an inner-product space X (e.g., $\mathbb{R}^d$), we denote the orthgonal complement of U by $U^\perp$, and the orthogonal projection of some $x \in X$ on a U by $\mathrm{Proj}_U(x)$.

II. Network Optimization and Passivity in Cooperative Control

The role of network optimization theory in cooperative control was introduced in [15]. In this section, we provide an overview of the main results from this letter.

A. Maximally Monotone Dynamical Systems

Consider the dynamical system of the form

$$\Upsilon : \left\{ \dot{x} = f(x, u),\ \ y = h(x, u), \right. \tag{1}$$

where $u \in \mathbb{R}$ is the input and $y \in \mathbb{R}$ is the output. In [18], the notion of *equilibrium independent passivity* (EIP) was introduced. EIP systems requires the existence of a continuous and monotone function that maps constant input signals to constant output signals, i.e., an equilibrium input-output map, and that the system is passive with respect to these input-output pairs. An extension of this notion proposed in [15] is *maximal equilibrium independent passivity* (MEIP). For MEIP, we consider the set of all pairs (u_{ss}, y_{ss}) of steady-state inputs and outputs for Υ and denote this by the *relation* k_Υ, which is a set of pairs of real numbers. Thus, the set of all the steady-state outputs associated with the input u is $k_\Upsilon(u)$, and the set of all steady-state inputs associated with the output y by $k_\Upsilon^{-1}(y)$. These are both set-valued maps, as their image can have more than one point, or no points at all. For example, if Υ is the simple integrator $\dot{x} = u,\ y = x$, then $k_\Upsilon = \{(0, y) : y \in \mathbb{R}\}$. This is the main distinguishing point between EIP and MEIP. In EIP, it is required that the steady-state input-output maps are *functions*, while for MEIP they should be *relations*. Consequently, the integrator example is not EIP.

Definition 1 (Maximal Equilibrium Independent Passivity [15]): Let Υ be as in (1). The system Υ is *maximally equilibrium independent monotonic (output-strictly) passive* if the following conditions hold:

i) The system Υ is (output-strictly) passive with respect to any steady state $(u_{ss}, y_{ss}) \in k_\Upsilon$ [19].

ii) The relation k_Υ is *maximally monotone*. That is, if $(u_1, y_1), (u_2, y_2) \in k_\Upsilon$ then either $(u_1 \le u_2$ and $y_1 \le y_2)$, or $(u_1 \ge u_2$ and $y_1 \ge y_2)$, and k_Υ is not contained in any larger monotone relation [20].

Such systems include (among others) simple integrators, gradient systems, Hamiltonian systems on graphs, and others (see [15] for more examples).

The main reason one is interested in monotone relations is their connection with convex functions. A theorem by Rockafellar [20] states that maximal monotone relations are given by the subdifferential of a convex function $\mathbb{R} \to \mathbb{R}$, and vice versa. Futhermore, this correspondence is unique up to a constant added to the convex function. In particular, for MEIP systems, we conclude that there exists some convex function K_Υ such that the steady-state input-output relation $k_\Upsilon(u)$ is the subgradient $\partial K_\Upsilon(u)$. This characteristic allows us to use the theory of convex optimization to find steady-states, as was exhibited in [15].

B. The Network Model

In this subsection, we describe the structure of the network dynamical system studied in [15]. Furthermore, we present the tool connecting the space of networked dynamical systems and network optimization theory.

Consider a collection of agents interacting over a network $\mathcal{G} = (\mathbb{V}, \mathbb{E})$. Assign to each node $i \in \mathbb{V}$ the dynamical system

$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h_i(x_i, u_i), \end{cases} \tag{2}$$

which we assume to be maximally equilibrium-independent monotonic passive. Similarly, we can assign a dynamical system (a controller[2]) to each edge $k \in \mathbb{E}$,

$$\Pi_k : \begin{cases} \dot{\eta}_k = \phi_k(\eta_k, \zeta_k) \\ \mu_k = \psi_k(\eta_k, \zeta_k), \end{cases} \tag{3}$$

also assumed to be maximally equilibrium-independent monotonic passive. We consider stacked vectors of the form $u = [u_1^T, \ldots, u_{|\mathbb{V}|}^T]^T$ and similarly for y, ζ and μ. The network system is diffusively coupled with the controller input described by $\zeta = \mathcal{E}^T y$, and the control input to each system by $u = -\mathcal{E}\mu$. This structure is illustrated in Fig. 1. In the figure, the systems Σ_i denote the agents, the systems Π_k denote the edge couplings, and $\mathcal{E}$ is the incidence matrix that provides the diffusive coupling between the nodal systems and edge controllers.

We denote the steady-state input-output relations of the node i and the edge k by k_i and γ_k, respectively. Owing to

[2]In the literature, controllers are usually external systems whose trajectory all the nodes in the network should be driven onto [21]. Here, the notion of controllers refers to edge couplings, which is also widespread in [15].

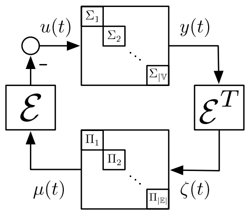


Fig. 1. Block-diagram of the closed loop.

Rockafellar's result, we also associate to each of the input-output relations the convex functions $K_i(u_i)$ and $\Gamma_k(\zeta_k)$ (i.e., $\partial K_i(u_i) = k_i$ and $\partial\Gamma_k(\zeta_k) = \gamma_k$). We consider the stacked relations $k(u)$ and $\gamma(\zeta)$ by concatenating the $k_i(u_i)$'s and $\gamma_k(\zeta_k)$'s respectively. We also define the convex functions $K(u) = \sum_{i\in\mathbb{V}} K_i(u_i)$ and $\Gamma(\zeta) = \sum_{k\in\mathbb{E}} \Gamma_k(\zeta_k)$. It is straightforward to check that $\partial K(u) = k(u)$ and $\partial\Gamma(\zeta) = \gamma(\zeta)$.

Remark 1: Although the defined network is unweighted, results obtained will also hold for weighted network, as one can "absorb" the weight in the measurement μ_k of each edge controller.

For the statement of the main theorem, we introduce the notion of the dual function. The dual function of K is defined by $K^\star(y) = \min_u\{y^T u - K(u)\}$ [22]. It is also a convex function, and it possesses the property that $\partial K^\star(y) = k^{-1}(y)$. One can similarly define the convex dual $\Gamma^\star(\mu)$ of Γ. We are now ready to state the main result from [15].

Theorem 1 [15]: Assume that the systems Σ_i are maximally equilibrium-independent monotonic output-strictly passive systems. Then the signals $u(t), y(t), \zeta(t), \mu(t)$ of the closed-loop system with diffusive coupling and controllers Π_k converge to some steady-state values $\hat{u}, \hat{y}, \hat{\zeta}, \hat{\mu}$. These values are the (primal-dual) solutions of the following pair of convex optimization problems:

Optimal Potential Problem	***Optimal Flow Problem***
$\min_{y,\zeta} \quad K^\star(y) + \Gamma(\zeta)$ $s.t. \quad \mathcal{E}^T y = \zeta$	$\min_{u,\mu} \quad K(u) + \Gamma^\star(\mu)$ $s.t. \quad \mu = -\mathcal{E}u.$

For the remainder of this letter, we make the following assumption on the agent dynamics.

Assumption 1: The agent dynamics in (2) are maximally equilibrium-independent monotonic passive.

Remark 2: Owing to the symmetry of the system in Fig. 1, we can consider networks of plants that are only maximally equilibrium-independent monotonic passive with controllers that are maximally equilibrium-independent monotonic output-strictly passive without changing the convergence result of Theorem 1.

III. Network Optimization for Controller Synthesis

Our goal is to design the controllers Π_k on the edges to achieve a desired output for the networked system. We specify the desired outputs in terms of the relative output $\mathcal{E}^T y(t)$, and we term this desired configuration a *formation*.

Definition 2: A *formation* is a vector $\zeta \in \mathbb{R}^{|\mathbb{E}|}$ which is in $\mathrm{Im}(\mathcal{E}^T)$. We say that a vector $y \in \mathbb{R}^{|\mathbb{E}|}$ has formation ζ if $\mathcal{E}^T y = \zeta$, and that a system converges to a formation ζ if its output converges to a vector y which has formation ζ.

Our overarching goal in this section is to solve the following problem, which deals with forcing a certain steady-state formation on the output of the agents.

Problem 1: Let $\zeta^\star$ be the desired formation. Find controllers Π_k on the edges such that the output $y(t)$ of the closed-loop system will converge a vector $y^\star$ such that $\mathcal{E}^T y^\star = \zeta^\star$, that is, $\lim_{t\to\infty} \mathcal{E}^T y(t) = \zeta^\star$.

The section is divided into three subsections. The first deals with forcing the desired formation to exist as a steady-state output. The second one deals with assuring that the system will globally asymptotically converge to the said formation. Finally, the last deals with the problem of different formations, namely how to construct a "formation reconfiguration scheme" allowing us to switch the steady-state formation of the closed-loop system by slightly augmenting the controller.

A. Achieving a Desired Steady-State

We assume the node dynamics Σ_i and network structure $\mathcal{G}$ is given and fixed. In particular, the input-output relations k_i, the integral functions K_i, their sum K, and the incidence matrix $\mathcal{E}$ are all given. However, the input-output relations of the controllers γ_k and their integral functions Γ_k are not specified and should be designed.

Suppose we choose our controllers to be maximally equilibrium-independent monotonic passive controllers. As before, we denote the stacked input-output relation of all the controllers by γ, and let Γ be the corresponding integral function. The result of Theorem 1 showed that the closed-loop system converges to an output y which solves the convex Optimal Potential Problem (OPP).

The outline to the solution to the Problem 1 is given by studying the minimizers of the optimization problem OPP. We first prove the following proposition.

Proposition 1: Let $\hat{y}$ be a fixed stacked output vector and let $\hat{\zeta} = \mathcal{E}^T\hat{y}$. The pair $(\hat{y}, \hat{\zeta})$ is a minimizer of OPP if and only if the inclusion

$$k^{-1}(\hat{y}) + \mathcal{E}\gamma(\hat{\zeta}) \ni 0 \tag{4}$$

holds.

Note that we ask 0 to be in the subdifferential set because it can have more than one value.

Proof: The network optimization problem OPP can be written as an unconstrained optimization problem in terms of the variable y alone. We ask to minimize $F(y) = K^\star(y) + \Gamma(\mathcal{E}^T y)$. This is a convex function of y, so it is minimized only where the zero vector lies in its subdifferential [22]. Thus, by subdifferential calculus (see [22]) we obtain

$$0 \in k^{-1}(\hat{y}) + \mathcal{E}\gamma(\mathcal{E}^T\hat{y}).$$

Plugging in $\hat{\zeta} = \mathcal{E}^T\hat{y}$ gives the desired criterion. ■

The main point of (4) is that if one tries to solve Problem 1, one must find some potential vector $y^\star$ such that $\mathcal{E}^T y^\star = \zeta^\star$ and $k^{-1}(y^\star) \in \mathrm{Im}(\mathcal{E})$. Thus, the question to be asked is if one can actually find such a vector $y^\star$.

Theorem 2: For every ζ, there exists a vector y such that $\mathcal{E}^T y = \zeta$ and $k^{-1}(y) \cap \mathrm{Im}(\mathcal{E}) \neq 0$.

Proof: We consider the function F which is the restriction of $K^\star$ on the set $Y = \{y : \mathcal{E}^T y = \zeta\}$. As F is a convex function defined on an affine subspace, it must have a minimum at some point $y \in Y$. Moreover, we know that the zero vector lies in $\partial F(y)$. In [23], it was shown that $\partial F(y) = \mathrm{Proj}_{\ker \mathcal{E}^T}(k^{-1}(y))$.

Thus, because we know that the zero vector is in the subgradient of F at y, we conclude that there is some vector $u \in k^{-1}(y)$ such that $\mathrm{Proj}_{\ker \mathcal{E}^T}(u) = 0$, which is the same as $u \in \ker(\mathcal{E}^T)^\perp = \mathrm{Im}(\mathcal{E})$. Thus we choose $y^\star = y$. ■

The following corollary shows that any formation ζ can be achieved as a steady-state solution.

Corollary 1: For every ζ, there exists some vector μ such that both $0 \in k^{-1}(y) + \mathcal{E}\mu$ and $\mathcal{E}^T y = \zeta$ hold. In particular, if $\gamma(\zeta) = \mu$, then (4) is satisfied, implying that (y, ζ) is a minimizer of OPP.

Proof: Take y to be the vector from Theorem 2. As $k^{-1}(y) \cap \mathrm{Im}(\mathcal{E}) \neq \emptyset$, we conclude that there is some vector μ such that $\mathcal{E}(-\mu) \in k^{-1}(\mu)$. This is equivalent to $0 \in k^{-1}(y) + \mathcal{E}\mu$. ■

B. Asymptotic Convergence and Uniqueness of a Steady-State

Recall that Problem 1 aims to achieve the formation $\zeta^\star$. We found a corresponding vector $y^\star$ such that $(y^\star, \zeta^\star)$ is a minimizer of OPP. We also know that the closed-loop system converges to a solution of this optimization problem (assuming that assumption 1 holds). However, we should note that network optimization problems (and in general, convex optimization problems) can admit multiple solutions. In this case, we cannot assure convergence of the system to the desired formation. However, if we show that $(y^\star, \zeta^\star)$ is the *unique* solution, we would assure convergence to the correct formation.

In this direction, we define a new function of the parameter ζ, replacing the y term in the target function:

Definition 3: The *minimal potential function* G is a function with domain $\mathrm{Im}(\mathcal{E}^T)$ having values in $\mathbb{R}$, and defined by

$$G(\zeta) = \min\{K^\star(y) \mid \mathcal{E}^T y = \zeta\}. \tag{5}$$

Theorem 3: Consider the closed-loop system in Fig. 1, and let $\zeta^\star$ be the desired formation vector. Suppose that the controllers are maximally equilibrium-independent monotonic output-strictly passive, and denote their steady-state input-output relations by γ_k and their integral convex functions by Γ_k. Suppose that the following conditions hold:

1) There exists some $y^\star$ such that $\mathcal{E}^T y^\star = \zeta^\star$.
2) The equation (4) holds for the pair $(y^\star, \zeta^\star)$.
3) For any edge $k \in \mathbb{E}$, Γ_k is strictly convex in a neighborood of ζ_k.

Then the closed-loop converges to the desired formation $\zeta^\star$. Furthermore, the output of the system converges to some $\tilde{y}$ satisfying $\mathcal{E}^T \tilde{y} = \zeta^\star$.[3]

Proof: If (4) is satisfied, then $(y^\star, \zeta^\star)$ is a minimizer of OPP. If we show all minimizers are of the form $(y, \zeta^\star)$ for some y, then the proof will be completed by Theorem 1.

The optimization problem OPP in the variables (y, ζ) is equivalent to the unconstrained minimization of $G(\zeta) + \Gamma(\zeta)$ in the variable ζ, meaning that $\zeta^\star$ is a minimizer for the latter. We show that it is the only minimizer. Indeed, we know that G is convex [23], and Γ is assumed to be convex and strictly convex in a neighborhood U of $\zeta^\star$. Hence, $G + \Gamma$ is a convex function which is strictly convex in U. Let M be the collection of all minima of $G + \Gamma$. It was shown in [23] and [22] that M is convex, and that $U \cap M$ contains no more than one point.

Now, we know that $\zeta^\star$ is a minimizer of $G + \Gamma$, so $\zeta^\star \in M$. If we had any other point $\zeta \in M$, the straight line from $\zeta^\star$ to ζ would have been in M, and in particular, since U is a neighborhood of $\zeta^\star$, it would contain infinitely many points from M. We saw that this cannot hold, meaning that $\zeta^\star$ must be the sole point in M, thus being the unique minimizer of $G + \Gamma$. This completes the proof. ■

Remark 3: In the general case, checking whether some function Γ is strictly convex (near $\zeta^\star$) is a hard task - one can try and show that for all vectors ζ_0, ζ_1, μ_0 and any $t \in \mathbb{R}$, the function $t \mapsto \mu_0^T \gamma(\zeta_0 + t\zeta_1)$ has no straight lines in its graph, which is not easily checked.

However, one should note that $\Gamma(\zeta) = \sum_i \Gamma_k(\zeta_k)$ has the property that it is strictly convex (near $\zeta^\star$) if and only if Γ_k are strictly convex near $\zeta_k^\star$ for all $k \in \mathbb{E}$. This is easier to verify geometrically since the $\Gamma_k : \mathbb{R} \to \mathbb{R}$ are one-dimensional maps. In particular, Γ_k is strictly convex if its graph does not contain any straight lines, or equivalently, γ_k does not have any horizontal lines for all $k \in \mathbb{E}$. We emphasize a few special cases of importance.

- The relation $\gamma(\zeta) = \nabla\psi(\zeta)$ defines a strictly convex function if and only if ψ is strictly convex.
- The relation $\gamma(\zeta) = M\zeta + \mu_0$ defines a strictly convex function if and only if M is a positive-definite function.
- If γ defines a convex relation and it is given by a differentiable map $\gamma(\zeta) = \phi(\zeta)$, then it defines a strictly convex function near $\zeta^\star$ if the differential $df(\zeta)$ is a full-rank matrix. Note that it is possible that this condition is violated, but that we still have a strictly convex function - for example $\gamma_k(\zeta_k) = \zeta_k^3$ with $\zeta^\star = \mathbf{0}$.

Example 1 (Linear Controllers): We consider an example for the construction of linear dynamic controllers of the form

$$\Pi_k : \begin{cases} \dot{\eta}_k = -\eta_k + \zeta_k + v_k \\ \mu_k = \eta_k, \end{cases}$$

for $k \in \mathbb{E}$, where the constant signals v_k are to be decided later. It is easy to check that this is a maximal equilibrium-independent monotonic output-strictly passive system with a steady-state input-output relation given by $\gamma_k(\zeta_k) = \zeta_k + v_k$. In this case, the stacked input-output relation is given by $\gamma(\zeta) = \zeta + v$, where v is freely-assignable.

Suppose we want a steady-state formation $\zeta^\star$. We first solve (4) and find some $y^\star$ such that $\mathcal{E}^T y^\star = \zeta^\star$ with some corresponding $u^\star \in k^{-1}(y^\star) \cap \mathrm{Im}(\mathcal{E})$. This is done by minimizing $K^\star(y)$ over $\{y : \mathcal{E}^T y = \zeta^\star\}$. Now take some μ such that $u^\star = \mathcal{E}(-\mu^\star)$. Then, (4) takes the form,

$$\mathcal{E}(-\mu^\star) = k^{-1}(y^\star) = \mathcal{E}(\zeta^\star + v),$$

[3]The vectors $\tilde{y}$ and $y^\star$ need not be identical.

and choosing $v = -\zeta^\star - \mu^\star$ satisfies (4). Moreover, the integral function Γ of γ is of the form $\Gamma(\zeta) = \frac{1}{2}\zeta^T\zeta + v^T\zeta$, which is strictly convex. Thus, if we take the controllers

$$\Pi_k : \left\{ \dot{\eta}_k = -\eta_k + \zeta_k - \zeta_k^\star - \mu_k^\star, \quad \mu_k = \eta_k \right.$$

we end up with the desired formation.

An important consequence of this result is that if all of the nodal systems are maximally equilibrium-independent monotone passive, then we can achieve any desired steady-state formation using a linear controller. However, one should note that the construction of these controllers requires global information, as the minimization of $K^\star(y)$ over $\mathcal{E}^T y = \zeta^\star$ is needed.

C. Formation Reconfiguration

In practical applications, we may want to change the desired formation $\zeta^\star$ after some time. However, we wish to avoid a change in the controller design scheme. Note that in the previous example, the desired formation $\zeta^\star$ defined the vector v we used. We wish to implement a similar mechanism for general controllers.

In this direction, we present a new scheme for the controller, adding a *constant* exogenous input ω_k. The controller dynamical system now becomes

$$\overline{\Pi}_k : \begin{cases} \dot{\eta}_k = \phi_k(\eta_k, \zeta_k, \omega_k) \\ \mu_k = \psi_k(\eta_k, \zeta_k, \omega_k). \end{cases} \tag{6}$$

This design allows us to alter the design of the system by changing ω_k, yielding different steady-state formations.

Suppose we have found some controllers of the form (3) for some formation ζ_0, and consider the stacked controller:

$$\Pi : \begin{cases} \dot{\eta} = \phi(\eta, \zeta) \\ \mu = \psi(\eta, \zeta). \end{cases}$$

We now augment it to a stacked new-scheme controller by allowing the exogenous input $\omega = (\alpha, \beta)$ to affect the output of the controller,

$$\overline{\Pi} : \begin{cases} \dot{\eta} = \phi(\eta, \zeta - \alpha) \\ \mu = \psi(\eta, \zeta - \alpha) + \beta. \end{cases}$$

The following result implies that it is enough to solve the synthesis problem for a single formation (e.g., concensus), applying the "formation reconfiguration" procedure to get any other desired formation.

Theorem 4: Consider the closed-loop system in Fig. 1 with some nominal controller Π, and suppose that its output converges to a formation ζ_0. Then there is a function $g : \zeta \mapsto \omega$ such that for any desired formation $\zeta^\star$, if one defines $\alpha = \zeta^\star - \zeta_0$ and $\beta = g(\zeta^\star) - g(\zeta_0)$, then the controller $\overline{\Pi}_\omega$ forces the output of the closed-loop system to converge to the formation $\zeta^\star$.

The controllers produced by the formation reconfiguration scheme is illustrated in Fig. 2.

Proof: The steady-state input-output relations can be computed from one another by $\overline{\gamma}_\omega(\zeta) = \gamma(\zeta - \alpha) + \beta$, where γ is Π's relation and $\overline{\gamma}_\omega$ is $\overline{\Pi}$'s relation for that specific choice of ω. For each ζ, we know that there is some y_ζ such that $\mathcal{E}^T y_\zeta = \zeta$ and that $\mathcal{E}(-\mu_\zeta) \in k^{-1}(y_\zeta)$ for some μ_ζ. Because Π yields the steady-state formation ζ_0, we know that $k^{-1}(y_{\zeta_0}) = -\mathcal{E}(\gamma(\zeta_0))$.

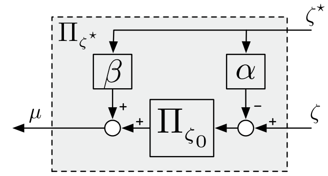


Fig. 2. The formation reconfiguration scheme.

Now, if we fix some formation $\zeta^\star$ and use $\overline{\Pi}$ with $\alpha = \zeta^\star - \zeta_0$ and $\beta = \mu_{\zeta^\star} - \mu_{\zeta_0}$, we claim that the equation (4) holds. Indeed,

$$\begin{aligned} k^{-1}(y_{\zeta^\star}) &= k^{-1}(y_{\zeta_0}) + (k^{-1}(y_{\zeta^\star}) - k^{-1}(y_{\zeta_0})) \\ &= -\mathcal{E}(\gamma(\zeta_0)) - \mathcal{E}(\mu_{\zeta^\star} - \mu_{\zeta_0}) \\ &= -\mathcal{E}(\gamma(\zeta_0) - \mu_{\zeta_0} + \mu_{\zeta^\star}) \\ &= -\mathcal{E}(\gamma(\zeta^\star - \alpha) + \beta) = -\mathcal{E}(\overline{\gamma}_\omega(\zeta^\star)). \end{aligned}$$

which proves our claim. ■

IV. Case Studies

A. Simple Integrators

We now focus on the case in which our agents are simple integrators. They are governed by the equations $\dot{x}_i = u_i;\ y_i = x_i$. The input-output steady-state of each node and the corresponding integral function is given by:

$$k_i(u_i) = \begin{cases} \mathbb{R}^N, & u_i = \mathbf{0} \\ \emptyset, & u_i \neq \mathbf{0} \end{cases}, \quad K_i(u_i) = \begin{cases} 0, & u_i = \mathbf{0} \\ \infty, & u_i \neq \mathbf{0}, \end{cases} \tag{7}$$

which has a dual function $K_i^\star(y_i) = 0$. This simplifies the problem OPP, as it reduces to optimizing $\Gamma(\zeta)$ over $\zeta \in \mathrm{Im}(\mathcal{E}^T)$. Equivalently, we can start from (4) and conclude that the equation at the minimum is just $\mathcal{E}\gamma(\zeta) = 0$.

Suppose we want to reach output agreement (i.e., the formation $\zeta^\star = 0$). We need $\gamma(0)$ to be in the kernel of $\mathcal{E}$. Thus a pick as $\gamma_k(\zeta_k) = \zeta_k \cdot \exp(\zeta_k^2)$ is viable. Furthermore, the corresponding convex function is $\Gamma_k(\zeta_k) = \frac{1}{2}\exp(\zeta_k^2)$, which is strictly convex. Thus the closed-loop system converges to a steady-state output-agreement. Implementing these controllers leads to the closed-loop system

$$\dot{x}_i = \sum_{j:(i,j)\in\mathbb{E}} (x_j(t) - x_i(t)) \cdot (\exp((x_j(t) - x_i(t))^2)),$$

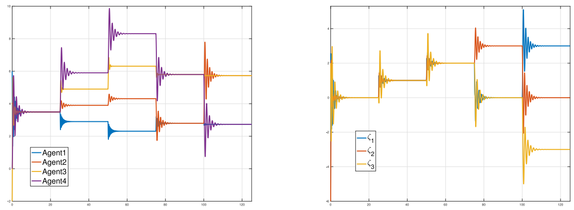
which is a nonlinear coupling driving the system to consensus.

Similarly, the choice $\gamma(\zeta) = \zeta$ with $\Gamma(\zeta) = \frac{1}{2}\|\zeta\|^2$, will lead to the well-known linear consensus protocol [4], [6].

B. Formation Reconfiguration of Damped Oscillators

We consider a network of four damped SISO oscillators,

$$\Sigma_i \left\{ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -b_i x_2 - \omega_i^2 (x_1 - \mathrm{x}_i) + u \end{bmatrix}, \ y = x_1 \right.$$



(a) Output of each oscillator. (b) The formation vector $\zeta(t) = \mathcal{E}^T y(t)$ of the oscillators.

Fig. 3. Formation control of damped oscillators.

where x_i is the equilibrium point of the spring. The underlying graph was chosen to be a path graph (i.e., $\mathbb{V} = \{v_1, v_2, v_3, v_4\}, \mathbb{E} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}$). For the diagonal matrix W with ω_i on the diagonal, the input-output steady-state relations is given by $k(u) = W^{-2}u + \mathrm{x}$. This implies that $K^\star(y) = \frac{1}{2}y^T W^2 y - y^T W^2 \mathrm{x}$, and the minimization algorithm solving (4) can be solved by methods of quadratic programming. For this example, the values ω_i, b_i, and x_i were chosen randomly as $\omega = [15.54,\ 5.13,\ 7.89,\ 4.29](Hz)$, $b = [1.66,\ 1.22,\ 4.62,\ 1.23](1/sec)$, $\mathrm{x} = [3,\ -2,\ 1,\ 0](m)$.

For a consensus objective, $\zeta^\star = \mathbf{0}$, (4) reduces to $\mathcal{E}\gamma(\zeta^\star) = 0$. Solving for this $\zeta^\star$, we choose $\gamma(\zeta) = \tanh(\zeta)$. To implement the said input-output relation, we take the following SISO controller on each of the edges,

$$\begin{cases} \dot{\eta}_k = -\eta_k + \zeta_k \\ \mu_k = \tanh(\eta_k). \end{cases}$$

We then use the formation reconfiguration scheme to create an augmented controller. The desired formation was changed every 25 seconds as $\zeta^1 = [0, 0, 0]^T$, $\zeta^2 = [1, 1, 1]^T$, $\zeta^3 = [2, 2, 2]^T$, $\zeta^4 = [0, 3, 0]^T$, and $\zeta^5 = [3, 0, -3]^T$. The output $y(t)$ of the system can be seen in Fig. 3(a) and relative outputs ζ in Fig. 3(b). We can see that the agents do as their supposed to, converging to the desired formations.

V. Conclusion

In this letter we presented a synthesis procedure for designing controllers in a networked system to guarantee convergence to a desired formation. We fixed a collection of MEICMP agents in an underlying graph, and derived the following results. First, we showed that any relative output vector ζ can be forced to be a steady-state of a closed-loop system, for some choice of couplings. Secondly, we showed that for any formation ζ, one can find edge controllers so that the closed-loop system globally asymptotically converges to the formation ζ. Thirdly, we gave criteria for the construction of the controllers in derived condition, and showed how to build controllers satisfying these criteria. Lastly, we presented the "formation reconfiguration" scheme, allowing to augment controllers driving the closed-loop system to some formation ζ_0, so that the closed-loop system with the augmented controllers will converge to any wanted formation. The results were demonstrated by two cases studies - simple integrators and damped oscillators.

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